Evolutionary games with random number of interacting players applied to access control

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Abstract

We study the interaction among wireless nodes and medium access control design in game theory framework. We define a general class of games, called medium access games, to capture the interaction among wireless nodes with medium access. We study an evolutionary Multiple Access Game (MAG) with a random number of nodes in interaction. We characterize Nash equilibria and evolutionary stable strategies (ESS) of MAG and strategy evolutions to achieve evolutionary stable strategies under replicator dynamics with delay. A new model of interaction between random number of nodes is proposed. We give examples of non-reciprocal interaction in evolutionary games and we optimize the probability of success of transmission of a node in the context on different node distribution in a plan which are the Dirac and the Poisson distributions.

Index Terms

multiple access game, evolutionary games, node distribution.

I. INTRODUCTION

The evolutionary game framework is used for modeling competition among large populations through many local interactions, each involving a small number of users. It introduces the concept of Evolutionary Stable Strategy (ESS), as well as the population dynamics that result from the interactions between the populations.

The ESS, first defined in 1972 by the biologist M. Smith [8], is characterized by a property of robustness against invaders (mutations). More specifically, (i) if an ESS is reached, then the proportions of each population do not change in time (ii) at ESS, the populations are immune from being invaded by other small populations. The ESS equilibrium concept is better adapted to large populations of players as it describes robustness against deviations of a whole fraction of the population as opposed to the Nash equilibrium concept that does not apply to deviations of more than a single player. We refer the reader to [4], [14], [3], [7], [2] for more details on ESS and evolutionary game dynamics.

Related works

Several previous papers have already studied evolutionary games with pairwise local interactions in the context of wireless networks. Bonneau et al. have introduced evolutionary games in the context of unslotted ALOHA in [1]. They have identified conditions for the existence of non trivial ESS and have computed them explicitly. In [9], the authors considered the multiple access game and studied delay effect under various models of evolutionary game dynamics with asymmetric delay based on the theoretic results on stability obtained in [10]. In [11], the authors extended this model by including a regret cost, incurred when no user transmits, and studied the impact of that cost on the proportion of mobiles that transmit at equilibrium. In the last three papers, the delay is shown to have negative impact on the stability of the system.

For other applications of evolutionary games concepts in networking, see [6], [15] who study congestion control models.

In this paper, we extend the evolutionary game framework to allow an arbitrary, possibly random, number of players that are involved in a local interaction; we apply this to the model of [11] which we extend to more than two interacting nodes. In the context of Medium Access Game, we study the impact of the node distribution in the game area on the equilibrium stable strategies of the evolutionary game. The interaction between more than two individuals in a population is a new concept in evolutionary game theory and has a lot of application in multiple access game in wireless networks. Considering this kind of games, we use the notion of expected utility as this game...
is not symmetric, indeed the number of players with which a given one interacts may vary from one to another; and also non-reciprocity property. We consider the following parameters in the multiple access game: transmission cost, collision cost and regret cost. Finally, we analyze the impact of these parameters on the probability of successful transmission and give some optimization issues.

The paper is structured as follows. We first provide in the next section an evolutionary game model with random number of opponent. In section II. We then study in section III a generalized multiple access game in the context of random number of players, we compute the expression of the ESS in this typical context. After that, we analyze in section IV the probability of success transmission. Numerical solutions of replicator dynamics are investigated in section IV-B.

II. EVOLUTIONARY GAMES WITH LOCAL INTERACTION AMONG RANDOM NUMBER OF PLAYERS

The classical evolutionary game formalism is a central mathematical tool developed by biologists for predicting population dynamics in the context of interaction between populations. In order to make use of the wealth of tools and theory developed in the biology literature, many works in the area of computer networks [10] ignore cases where local interactions between populations involve more than two individuals. This restriction limits the modeling power of evolutionary games which are not useful in a network operating at heavy load, such as ad-hoc networks with high density (see section IV). This motivated us in this paper to consider a random number of users interacting locally.

Consider a large population of players. Each individual needs occasionally to take some action. When doing so, it interacts with the actions of some \( M \) (possibly random number of) other individuals.

A. Symmetry and Reciprocity

We shall consider throughout the paper a symmetric game in the sense that any individual faces the same type of game. All players have the same actions available, and same expected utility. We note however that the actual realizations need not be symmetric. In particular, (i) the number of players with which a given player interacts may vary from one player to another. (ii) We do not even need the reciprocity property: if player A interacts with player B, we do not require the converse to hold. We provide some examples of multiple access games to illustrate this non-reciprocity.

For example, we consider local interactions between transmitters; for each transmitter there corresponds a receiver. We shall say that a transmitter A is subject to an interaction (interference) from transmitter B if the transmission from B overlaps that from A, and provided that the receiver of the transmission from A is within interference range of transmitter B.

**Example 1** Consider the example depicted at Figure 1. It contains 4 sources (circles) and 3 destinations (squares). A transmission of a source \( i \) within a distance \( r \) of the receiver \( R \), causes interference to a transmission from a source \( j \neq i \) to receiver \( R \). We see that Source A and Source C cause no interference to any other transmission but the transmission from A suffers from interference from source B, and the one from C suffers from the transmission of the top most source (called D). Source B and D interfere with each other at their common destination. Thus each of the four sources suffers interference from a single other source, but except for nodes B and D, the interference is not reciprocal.

![Fig. 1. Non-reciprocal pairwise interactions](image-url)
Example 2. In Figure 2 there are four sources and only two destinations. Node A does not cause any interference to the other nodes but suffers interference from nodes B and D. Nodes B, C, D interfere with each other. This is a situation in which each mobile is involved in interference from two other mobiles but again the interference is not reciprocal.

![Fig. 2. Non-reciprocal interactions between groups three players](image1)

![Fig. 3. Interactions between a random number of players](image2)

Example 3. In this example the number of interfering nodes is not fixed. A suffers interference from 2 nodes, B and D suffer interference from a single other node and C does not suffer (and does not cause) interference. All examples exhibit asymmetric realizations and non-reciprocity. We next show how such a situation can still be considered as symmetric (due to the fact that we consider distributions of nodes rather than realizations). Assume that the location of the transmitters follow a Poisson distribution with parameter $\lambda$ over the two dimensional plane. Consider an arbitrary user A. Let $L$ be the interference range. Then the number of transmitters within the interference range of the receiver of A has a Poisson distribution with parameter $\lambda \pi L^2/2$. Since this holds for any node, the game is considered to be symmetric. The reason that the distribution is taken into account rather than the realization is that we shall assume that the actions of players will be taken before knowing the realization.

B. Model

We describe in this part notations of our model.

- There is one population of users. The number of users is large.
- Each member of the population chooses from the same set of strategies $\{1, 2, \ldots, N\}$.
- Let $M := \{x = (x_1, \ldots, x_N) \mid x_j \geq 0, \sum_{j=1}^{N} x_j = 1\}$ the set of probability distributions over the $N$ strategies. $M$ can be interpreted as the set of mixed strategies. It is also interpreted as the set of distributions of strategies among the population, where $x_j$ represents of proportion of users choosing the strategy $j$. A distribution $x$ is sometime called the "state" or "profile" of the game.
- The number of opponents $K$ of a user is a random variable in the finite set $\{0, 1, \ldots, k_{\text{max}}\}$. $k_{\text{max}}$ is the maximum number of opponents interacting simultaneously with a user. We assume that we can ignore cases of interaction with more than $k_{\text{max}}$ players. This value depends on the node density and the transmission range. When making a choice of a strategy, a player knows the distribution of $K$ but not its realization.
- The payoff of all players functions (identical for each member of the population) of the player’s own behavior and opponents’ behavior. Each user’s payoff depends on opponents’ behavior through the distribution of opponents’ choices and of their number. The expected payoff of a user playing strategy $j$ when the state of the population is $x$, is given by

$$f_j(x) = \sum_{k=0}^{k_{\text{max}}} P(K = k) u_k(j, x, \ldots, x), \quad j = 1, \ldots, N$$

where $u_k$ is the payoff function given that the number of opponents is $k$. Although the payoffs are symmetric, the actual interference or interactions between two players that use the same strategy need not be the same, allowing for non-reciprocal behavior. The reason is that the latter is a property of the random realization whereas the actual payoff already averages over the randomness related to the interactions, the number of interfering players, the topology etc.
C. Evolutionary Stable Strategies: ESS

Suppose that, initially, the population profile is \( x \in M \). The average payoff is

\[
f(x, x) = \sum_{j=1}^{N} x_j f_j(x).
\] (1)

Now suppose that a small group of mutants enters the population playing according to a different profile \( \text{mut} \in M \). If we call \( \epsilon \in (0, 1) \) the size of the subpopulation of mutants after normalization, then the population profile after mutation will be \( \epsilon \text{mut} + (1- \epsilon)x \). After mutation, the average payoff of non-mutants will be given by \( \epsilon f(x, \text{mut}) + (1- \epsilon)f(x, x) \) where

\[
f(x, \text{mut}) = \sum_{j=1}^{N} x_j f_j(\text{mut}).
\]

Analogously, the average payoff of a mutant is \( (1- \epsilon)f(\text{mut}, x) + \epsilon f(\text{mut}, \text{mut}) \).

**Definition 1:** A strategy \( x \in M \) is an ESS if for any \( \text{mut} \neq x \), there exists some \( \epsilon_{\text{mut}} \in (0, 1) \), which may depend on \( \text{mut} \), such that for all \( \epsilon \in (0, \epsilon_{\text{mut}}) \)

\[
\epsilon f(x, \text{mut}) + (1- \epsilon)f(x, x) > (1- \epsilon)f(\text{mut}, x) + \epsilon f(\text{mut}, \text{mut})
\] (2)

That is, \( x \) is ESS if, after mutation, non-mutants are more successful than mutants. In other words, mutants cannot invade the population and will eventually get extinct.

Equation (2) may be rewritten as

\[
\epsilon (-f(\text{mut, mut}) + f(x, \text{mut})) + (1- \epsilon) (f(x, x) - f(\text{mut}, x)) > 0.
\] (3)

There is close a relation between ESS and Nash equilibrium of the following matrix-game

\[
\Gamma := (\{1, 2, \ldots, k_{\text{max}}\}, \{T, S\}, r(\cdot))
\]

where \( r : \{T, S\}^{k_{\text{max}}} \rightarrow \mathbb{R}, r(a_1, a_2, \ldots, a_{k_{\text{max}}}) = \sum_{l=1}^{k_{\text{max}}} u_l(a_1, a_2, \ldots, a_l)\mathbb{P}(K = l) \).

It is easy to see that inequality (3) is equivalent to the two following conditions:

- Nash equilibrium condition of the matrix-game \( \Gamma \).
  \[
  \forall \text{mut} \in M, f(\text{mut}, x) \leq f(x, x),
  \] (4)

- Stability condition
  \[
  \text{if mut} \neq x, \text{ and } f(\text{mut}, x) = f(x, x) \text{ then } f(\text{mut, mut}) < f(x, \text{mut}).
  \] (5)

D. Stability of the Replicator dynamics

Replicator dynamics is one of the most studied dynamics in evolutionary game theory. The replicator dynamics has been used for describing the evolution of road traffic congestion in which the fitness is determined by the strategies chosen by all drivers [12]. It has also been studied in the context of the association problem in wireless networks in [13]. We introduce the replicator dynamics which describes the evolution in the population of the various strategies. In the replicator dynamics, the share of a strategy \( j \) in the population grows at a rate proportional to the difference between the payoff of that strategy and the average payoff of the population. The replicator dynamic equation is given by

\[
\dot{x}_j(t) = \mu x_j(t) \left[ f_j(x(t)) - \sum_{k=1}^{N} x_k f_k(x(t)) \right].
\] (6)

The parameter \( \mu \) doesn’t change the evolutionary stable strategies set but have a big influence on the stability of the system. We obtain the classical model of evolutionary game when the random variable \( K \) is the Dirac distribution of the number two. We will discuss in the next section about the case \( K = \delta_{\{n-1\}} \) where \( \delta_{\{n-1\}} \) is the Dirac distribution concentrated in \( n-1 \).
III. MULTIPLE ACCESS GAME

The static multiple access game considered here is a generalization of the random access game considered by Inaltekin and Wicker in [5]. Multiple Access Game introduces the problem of medium access. We assume that mobiles are randomly placed over a plane. All mobiles use the same fixed transmission range of $r$. The channel is ideal for transmission and all errors are due to collision. A mobile decides to transmit a packet or not to transmit to a receiver when they are within transmission range of each other. Interference occurs as in the ALOHA protocol: if more than one neighbors of a receiver transmit a packet at the same time then there is a collision. The Multiple Access Game is a nonzero-sum game, the mobiles have to share a common resource, the wireless medium, instead of providing it. We suppose that a mobile has a receiver in its range with probability $\mu$. When a mobile $i$ transmits to its receiver $R(i)$, all mobiles within a circle of radius $r$ centered at the receiver $R(i)$ cause interference to the node $i$ for its transmission to receiver $R(i)$. This means that more than one transmission within a distance $r$ of the receiver in the same slot brings a collision of all packets at the receiver.

Each of the mobiles has two possible strategies: either to transmit ($T$) or to stay quiet ($S$). If mobile $i$ transmits a packet, it incurs a transmission cost of $\delta \geq 0$. The packet transmission is successful if the other users don’t transmit (stays quiet) in that given time slot, otherwise there is a collision and the cost (of collision risk) is $\Delta \geq 0$. If there is no collision, user $i$ gets a reward of $V$ from the successful packet transmission. We suppose that the reward $V$ is greater than the cost of transmission $\delta$. When all users stay quiet, they have to pay a regret cost $\kappa$. The regret cost $\kappa$ describes the behavior of mobiles when they are aware of the backoff delays. If $\kappa = 0$ the game is called degenerate multiple access game. In the figure 4, we describe an example of interaction of three nodes. The ESS corresponding to any number of nodes of this game is given in theorem 1 in Subsection ??.

![Table](image)

**Remark 1:** Consider that all nodes are randomly distributed on a plane following a Poisson point process with density $\lambda$ and all nodes are the same fixed transmission range of $r$. Hence the probability that the number of neighbors of a node is given by

$$P(K = k) = \frac{(\lambda \pi r^2)^k}{k!} \exp\{-\lambda \pi r^2\}$$

We observe that for $\lambda = 0.1$ and $r = 1$, the $P(K = 5) = 1.8626 * 10^{-5}$. In that case, we can ignore the interaction more than four users, i.e., $k_{max} = 4$. But, for $\lambda = 0.7$ and $R = 1$, we have $P(K = 5) = 0.0475$ and $P(K = 10) = 8.0845 * 10^{-5}$. In that case, we can take $k_{max} = 10$. However, the value $k_{max}$ depends on $\lambda$ and $r$.

Let $A := \{T, S\}$ be the set of strategies and denote by $x$ the population share of strategy $T$. We assume that the nodes policy reflects the state of the population i.e., anonymous opponents use the strategy $(x, 1-x)$ in the $(K+1)$-th matrix game. The payoff obtained by a node-mutant with $k$ opponents when it plays $T$ is

$$u_k(T, x) = \mu \left((-\Delta - \delta) (1 - \eta_k) + (V - \delta) \eta_k\right)$$

where $\eta_k := (1-x)^k$, and the node-mutant receives $u_k(S, x) = -\mu \kappa (1-x)^k$ when it stays quiet. The expected payoff of an anonymous transmitter node-mutant is given

$$f(T, x) := \mu \sum_{k=0}^{k_{max}} P(K = k) u_k(T, x) = \mu \left((-\Delta + \delta) + (V + \Delta) \sum_{k=0}^{k_{max}} P(K = k) (1-x)^k\right)$$

Analogously, we have

$$f(S, x) := \mu \sum_{k=0}^{k_{max}} P(K = k) u_k(S, x) = -\mu \kappa \sum_{k=0}^{k_{max}} (1-x)^k P(K = k).$$

(7)

From equation 1, the expected payoff of any individual in the population where $x$ is the proportion of mobiles which transmit, is given by:

$$f(x, x) = x f(T, x) + (1-x) f(S, x).$$

(8)
We next introduce three alternative information scenarios that have an impact on the decision making. In the first case, a mobile does not know whether there are zero or more other mobiles that interact with him. In the second case the mobile has this information, and consequently, in a first case, he decides to always transmit when he has no interferers; and in the second case, we consider always the interference game, each mobile is always in competition with other mobiles. We observe that the two first scenario give similar results but not the last scenario.

1) Case 1: without information: We consider in that case that a transmitter never knows if it is in competition with another mobile or not. Then, even when it is the only transmitter, it has to decide to transmit or not.

**Theorem 1:** If \( \mathbb{P}(K = 0) < \frac{\Delta + \delta}{V + \Delta + \kappa} =: \alpha \), then the game has a unique ESS \( x_1^* \) given by

\[
x_1^* = g^{-1}\left(\frac{\Delta + \delta}{V + \Delta + \kappa}\right) \quad \text{where} \quad g : \ x \mapsto \sum_{k=0}^{k_{\text{max}}} \mathbb{P}(K = k)(1 - x)^k.
\]

**Proof:** A mixed equilibrium \( x \) is characterized by \( f(T, x) = f(S, x) \) i.e

\[
g(x) = \frac{\Delta + \delta}{V + \Delta + \kappa}.
\]

The function \( g \) is continuous and strictly decreasing monotone on \((0, 1)\) with \( g(1) = \mathbb{P}(K = 0) \) and \( g(0) = 1 \). Then the equation (9) has a unique solution in \((\mathbb{P}(K = 0), 1)\). One has,

\[
f(x, \text{mut}) - f(\text{mut, mut}) = \mu(V + \Delta + \kappa)(s - \text{mut})(g(\text{mut}) - g(x)).
\]

Thus, \( f(x, \text{mut}) - f(\text{mut, mut}) > 0 \) (because \( g \) is strictly decreasing function) for all \( \text{mut} \neq x \). This completes the proof.

When a mobile is never alone in his interference area, i.e., \( \mathbb{P}(K = 0) = 0 \); the condition \( \alpha > 0 \) is satisfied and the game has an unique ESS described in theorem 1.

2) Case 2: with information and always transmit: In that second case, we consider that a mobile knows when it is the only one transmitter, and then he decides always to transmit in that configuration. We next give another result about the existence and uniqueness of an ESS in that case.

**Theorem 2:** Suppose that the action set is \( \{\text{transmit}\} \) for every user without opponents. Thus, an anonymous user without opponents receives the fitness \( f_0 = V - \delta \). If \( \mathbb{P}(K = 0) < \frac{\Delta + \delta}{V + \Delta + \kappa} \), then the game has a unique ESS \( x_2^* \) which given by

\[
x_2^* = g^{-1}\left(\frac{\Delta + \delta + \kappa\mathbb{P}(K = 0)}{V + \Delta + \kappa}\right) \quad \text{where} \quad g : \ x \mapsto \sum_{k=0}^{k_{\text{max}}} \mathbb{P}(K = k)(1 - x)^k.
\]

3) Case 3: never alone: In this last case, we consider that a mobile is never alone to transmit in a slot. Then, we suppose that \( \mathbb{P}(K = 0) = 0 \).

**Theorem 3:** The game has always an unique ESS which is solution of the equation \( \sum_{k=1}^{k_{\text{max}}} \mathbb{P}(K = k)(1 - s)^k = \alpha \) with \( \sum_{k=1}^{k_{\text{max}}} \mathbb{P}(K = k) = 1 \).

**Proposition 1:** The ESS given in theorem 1 is asymptotically stable under the replicator dynamics.

**Proof:** The replicator dynamics is given by

\[
\dot{x} = (V + \Delta + \kappa)x(1 - x)(g(x) - \alpha).
\]

The function \( g \) is decreasing on \((0, 1)\) implies that the derivative of the function \( x(1 - x)(g(x) - \alpha) \) at the ESS is negative. Hence, the ESS is asymptotically stable.

**IV. NODES DISTRIBUTION**

In this section, we consider different node distributions and their implication of the existence of an ESS and on the stability of the replicator dynamics. First, one, we assume that all mobiles have the same number of neighbors \( n - 1 \), i.e., \( \mathbb{P}(K = j) = \delta_{n-1}(j) \) and seconde one, we assume that nodes are randomly distributed on a plan following a Poisson point process with density \( \lambda \).

We denote the following cost ratio \( \alpha := \frac{\Delta + \delta}{V + \Delta + \kappa} \), which represents the ration between the collision cost \( -\Delta - \delta \) (cost when there is a collision during a transmission) and the difference between total cost perceived by a mobile \( -\Delta - \delta - \kappa \) (collision and regret) and the benefit \( V - \delta \) (reward minus transmission cost).
A. Dirac distribution: \( \mathbb{P}(K = j) = \delta_{n-1}(j) \)

In this part, we suppose that the population of nodes is composed with many local interaction between \( n \) nodes. Let \( A := \{T, S\} \) the set of strategies and assume that the strategy \( T \) has a delay \( \tau_T \) and the strategy \( S \) has the delay \( \tau_S \). The payoff of a player using the action \( a_i \in A \) against the other players when they use the multi-strategy \( a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \) is given by \( U_i(a) \).

Each user plays the \( n \)-player following game \( \Gamma_n = (N, A, (U_i)_{i \in N}) \) where
- \( N \) is the set of users (nodes) and the cardinal of \( N \) is \( n \),
- \( A \) the set of pure actions (the same for every user),
- for every user \( i \) in \( N \), the payoff function \( U_i : A^n \rightarrow \mathbb{R} \) is given by

\[
U_i(a) = \begin{cases} 
V - \delta & \text{if } a_i = T \text{ and } a_j = S \text{, } \forall j \neq i \\
0 & \text{if } a_i = S \text{ and } \{ j \in N \mid a_j = T \} \geq 1 \\
-\Delta - \delta & \text{if } a_i = T \text{ and } \{ j \in N \mid a_j = T \} \geq 2 \\
-\kappa & \text{if } a_j = S \text{, } \forall j \in N
\end{cases}
\]

Let \( x \) be the proportion of nodes in the population using the strategy \( T \). Then \((x, 1-x)\) is the state of the population. Let \( \Delta(A) := \{ xT + (1-x)S \mid 0 \leq x \leq 1 \} \) the set of mixed strategies. The average payoff is

\[
f(x, x) = \mu x \left[ (-\Delta - \delta) \left(1 - (1-x)^{n-1}\right) + (V - \delta) (1-x)^{n-1} \right] - \mu \kappa (1-x)^n.
\]

Note that the first interference scenario described in the previous section holds here because the number of interferes is fixed and is equal to \( n-1 \). Then from theorem 1 with the function \( g(x) = (1-x)^{n-1} \), the ESS exists and is uniquely defined by

\[
x^* = 1 - \alpha \frac{1}{x_{\alpha_n}^2}.
\]

Note that it is shown in [11] that when there are pairing local interaction in a single population, the completely mixed Nash equilibrium of the two nodes stage game \( \Gamma_2 \) is an ESS.

B. Poisson distribution

Here, we consider that nodes are distributed over a plan following a Poisson distribution with density \( \lambda \). The probability that a node has \( i \) neighbors is given by the following distribution. Cases 1 and 2:

\[
\mathbb{P}(K = k) = \frac{(\lambda \pi r^2)^k}{k!} e^{-\lambda \pi r^2}, \quad k \geq 0.
\]

Case 3 when a user is never alone:

\[
\mathbb{P}(K = k) = \frac{(\lambda \pi r^2)^{k-1}}{(k-1)!} e^{-\lambda \pi r^2}, \quad k \geq 1.
\]

Let \( A := \{T, S\} \) the set of strategies. From theorem 1, the unique ESS is solution of the following equation:

\[
\begin{cases} 
e^{-\lambda \pi r^2 x} = \alpha & \text{for case 1} \\
e^{-\lambda \pi r^2 x} = \frac{\Delta + \delta + \kappa \mathbb{P}(K=0)}{\Delta + \kappa} & \text{for case 2} \\
(1-x)e^{-\lambda \pi r^2 x} = \alpha & \text{for case 3}
\end{cases}
\]

Thus we obtain the following equilibria in the different scenarii:

\[
x^*_1 = \log \left( \alpha_1^{-\frac{1}{\lambda \pi r^2}} \right), \quad x^*_2 = \log \left( \alpha_2^{-\frac{1}{\lambda \pi r^2}} \right) \quad \text{and} \quad x^*_3 = 1 - \frac{\text{LambertW}(\lambda \pi r^2 \alpha_3 e^{\lambda \pi r^2})}{\lambda \pi r^2},
\]

with \( \text{LambertW}(x) \) is the Lambert W function which is the inverse function of \( l(w) = we^w \).

Convergence to the ESS under replicator dynamics (without delay)

In the figures 5,6 and 7 we compare the evolution of the fraction of transmitters varying the parameter of density \( \lambda \) between 0.1 and 5 for the case 1, 2 and 3 respectively. In all these figures, the replicator dynamic is without delays.

Figure 8 compare the three cases for \( \lambda = 0.5 \). The case 1 prime(resp. 2 prime) represents the proportion of transmitters with at least one opponent in the case 1 (resp. case 2).

V. Optimization issues

We look for the probability of success that can be achieved in a local interaction depending on distribution parameters and also cost parameters. First we consider the dirac distribution with parameter \( n \) the number of nodes in interaction. Second, we consider the Poisson distribution with parameters \( \lambda \) and \( L \).
Fig. 5. Evolution of the fraction of transmitters varying the density parameter $\lambda$ in the case 1(without delays).

Fig. 6. Evolution of the fraction of transmitters varying the density parameter $\lambda$ in the case 2(without delays).

A. Dirac distribution

At the equilibrium point, the probability of success of a node is given by $x^*(1-x^*)^{n-1}$ and the total probability to have a successful transmission in a local interaction (total throughput) is given by

$$P_{\text{succ}}(\alpha, n) = n\mu x^*(1-x^*)^{n-1} = n\mu(1 - \alpha^{\frac{1}{n-1}})\alpha,$$

(11)

where $\mu$ is the probability that a mobile has a receiver in its range.

Proposition 2: The probability to have a successful transmission $P_{\text{succ}}(\alpha, n) = n\mu(1 - \alpha^{\frac{1}{n-1}})\alpha$ goes to $-\mu\alpha \log(\alpha)$ when the number of nodes goes to infinity. $[0, \frac{1}{\mu}]$ Hence, when $n$ is large, the maximum total throughput of a local interaction is obtained when $\alpha = \frac{1}{e}$ and is closed to $\mu/e$.

We observe on figure 11, the total throughput depending on the number of interferes $n$. The variables considered are $\mu = 0.8$ and $\alpha = 1/3$. We denote that the total throughput is increasing in that case with the number of interferes which it seems non intuitive. The reason is that the number of transmitted mobiles at the ESS, i.e. $x^*$, is exponentially decreasing with $n$. Another important result is that it may have a finite number of interferers that maximize the total throughput like in figure 10.

Proposition 3: We fixed the number of interferes $n$. The optimal total throughput is obtained when

$$\alpha^* = (1 - \frac{1}{n})^{n-1}$$

and the corresponding throughput is

$$P_{\text{succ}}(\alpha^*, n) = \mu(1 - \frac{1}{n})^{n-1}.$$

In figure 13, we observe the total throughput for different value of $n$ and applying the result of the previous proposition, we obtain the optimal total throughput depending on $\alpha$. 
B. Poisson distribution

The probability to have a successful transmission in a local interaction (total throughput) is given by different equation depending on the scenario. In the case 1 we have:

\[
P_{\text{succ}}(\alpha_1, \lambda) = \mu x^*_1 \sum_{k=0}^{\lambda \pi r^2} k \mathbb{P}(K = k)(1 - x^*_1)^k = \mu x^*_1 \sum_{k=0}^{\lambda \pi r^2} \frac{(\lambda \pi r^2)^k}{k!}(1 - x^*_1)^k \approx \mu x^*_1 (1 - x^*_1) \lambda \pi r^2 \alpha_1,
\]
with $x_1^* = \log \left( \frac{1}{\alpha_1 - \lambda \pi r^2} \right)$. In the case 2, we have:

$$P_{\text{succ}}(\alpha_2, \lambda) \approx \mu x_2^* (1 - x_2^*) \lambda \pi r^2 \alpha_2$$

where $s_2^* = \log \left( \frac{1}{\alpha_2 - \lambda \pi r^2} \right)$.

It is easy to see that the following proposition holds

**Proposition 4:** The maximum of total throughput under poisson distribution is attained when $\alpha_1 = e^{h(\lambda, r)}$ in the case 1 (resp. $\alpha_2 = e^{h(\lambda, r)}$ in the case 2) where $h$ is one of the two functions defined by

$$(\lambda, r) \in \mathbb{R}_+^2 \mapsto \frac{-(1 + 2 \lambda \pi r^2) \pm \sqrt{1 + 4(\lambda \pi r^2)^2}}{2}$$

In the case 3, we have:

$$P_{\text{succ}}(\alpha, \lambda) = \mu x_3^* \sum_{k=1}^{k_{\text{max}}} k P(K = k)(1 - x_3^*)^k = \mu x_3^* \sum_{k=1}^{k_{\text{max}}} k \left( \frac{\lambda \pi r^2}{(k-1)!} \right) (1 - x_3^*)^k$$

, \approx \mu \alpha x_3^* (1 + \lambda \pi r^2 (1 - x_3^*)) ,

with $x_3^* = 1 - \frac{\text{LambertW}(\lambda \pi r^2 \alpha e^{(\lambda \pi r^2)})}{\lambda \pi r^2}$.

**Proposition 5:** There exists a unique $\alpha_3^*$ in which the total throughput is maximum when $\alpha = \alpha_3^*$. The $\alpha_3^*$ is given by

$$\alpha_3^* = (1 - x)e^{-\lambda \pi r^2 x}$$
where $x$ is the unique solution in $[0,1]$ of the following equation:

$$1 + \gamma - x(2 + 5\gamma + \gamma^2) + x^2(4\gamma + 2\gamma^2) - \gamma^2x^3 = 0$$

**Proof:** The derivative of the function $H := \frac{\partial P_{\text{succ}}}{\partial s} s$ given by

$$H(s) = (1 + \gamma - s(2 + 5\gamma + \gamma^2) + s^2(4\gamma + 2\gamma^2) - s^3\gamma^2)e^{-\gamma s}.$$  

We prove that the above function is strictly decreasing on $[0,1]$. For that, it is sufficient to study the following function

$$G(s) = 1 + \gamma - s(2 + 5\gamma + \gamma^2) + s^2(4\gamma + 2\gamma^2) - s^3\gamma^2.$$  

We have $\frac{\partial G(s)}{\partial s}$ is given by

$$\frac{\partial G(s)}{\partial s} = -(2 + 5\gamma + \gamma^2) + 2s(4\gamma + 2\gamma^2) - 3s^2\gamma^2.$$  

It is easy to show that the above function is always negative. Since $H(0) = 1 + \gamma > 0$ and $H(1) = -e^{-\gamma} < 0$ then the function $H$ is positive for $s \in [0,\bar{s})$ and is negative for $s \in (\bar{s},1]$ where $\bar{s}$ is the solution of the equation $G(s) = 0$. Since $s^*$ is decreasing function on $\alpha$, we conclude that function $P_{\text{succ}}$ is positive if $s \in [0,\bar{s})$ and is negative $s \in (\bar{s},1]$. Since the optimal of function $P_{\text{succ}}$ is attained at $\alpha = (1 - \bar{s})e^{-\lambda\pi r\bar{s}}$.  

The probability of success at the ESS in poisson distribution is represented in figures 14 and 15. We observe in particular case that, as in the Dirac distribution case, when the number of interferes increases, i.e. the rate $\lambda$ in the case of the Poisson distribution, the total throughput increases.

![Fig. 14. Probability of success in poisson distribution(cases 1,2).](image1)

![Fig. 15. Probability of success in poisson distribution(case 3).](image2)

**C. Replicator dynamics with delay**

In the replicator dynamics with delay, the share of a strategy $j$ in the population grows at a rate proportional to the difference between the payoff of that strategy delayed by a constant time $\tau_j$ and the average delayed payoff of the population. The replicator dynamic equation is given by

$$\dot{x}_j(t) = \mu x_j(t) \left[ f_j(x(t - \tau_j)) - \sum_{k=1}^N x_k f_k(x(t - \tau_k)) \right].$$  

(12)

The parameters $\tau_j$ and $\mu$ don’t change the evolutionary stable strategies set but have a big influence on the stability of the system.

1) **Dirac distribution case:** The replicator dynamic equation given in (12) becomes

$$\dot{s}(t) = \mu s(t)(1 - s(t)) \left[ f(T, s(t - \tau_T)) - f(S, s(t - \tau_S)) \right]$$  

(13)

where

$$f(T,s(t)) := -\mu(\Delta + \delta) (1 - (1 - s(t))^{n-1}) + \mu(\Delta - \delta)(1 - s(t))^{n-1}$$

and

$$f(S,s(t)) := -\mu\kappa (1 - s(t))^{n-1}.$$
In order to study the asymptotically stability of the replicator dynamics (13) around the unique ESS $s^*_1 = 1 - \left( \frac{\Delta + \delta}{V + \Delta + \kappa} \right)^{\frac{1}{r}},$ we linearize (13) at $s^*_1.$ We obtain the following linear delay differential equation

$$\dot{x}(t) = -\mu(n-1)s(1-s)^{n-1}((V + \Delta)x(t - \tau_T) + \kappa x(t - \tau_S))$$

(14)

where $x(t) = s(t) - s^*_1.$ The following theorem gives sufficient conditions of stability of (14) at zero.

**Theorem 4 (see [10]):** Suppose at least one of the following conditions holds

- $(V + \Delta)\tau_T + \kappa \tau_S < \frac{1}{n-1}s(1-s)^{n-1}$
- $(V + \Delta)\tau_T < \frac{(V + \Delta + \kappa)}{n-1}s(1-s)^{n-1}$
- $(V + \Delta)\tau_T < \frac{V + \Delta + \kappa}{n-1}s(1-s)^{n-1}$

Then the ESS $s$ is asymptotically stable.

A necessary and sufficient condition of stability of (14) at zero when delays are symmetric is given in theorem 5.

**Theorem 5 (symmetric delay):** Suppose that $\tau_T = \tau_S = \tau,$ then, the ESS $s^*_1$ is asymptotically stable if and only if

$$\tau < \frac{\pi}{2(n-1)\mu s^*_1(1-s^*_1)^{n-1}(V + \Delta + \kappa)}$$

The proof uses the following well known lemma (see [9]) and the references therein.

**Lemma 1:** The trivial solution of the linear delay differential equation

$$\dot{z}(t) = -az(t - \tau), \quad \tau, a > 0$$

is asymptotically stable if and only if $2a\tau < \pi.$

2) Poisson distribution case: The replicator dynamic equation given in (12) becomes

$$\dot{s}(t) = \mu s(t)(1-s(t)) [f(T, s(t - \tau_T)) - f(S, s(t - \tau_S))]$$

(15)

and

$$f(T, s(t)) := \mu \left( -\Delta + \delta \right) + (V + \Delta)e^{-\lambda \pi r^2 s}$$

and $f(S, s(t)) := -\mu s^* e^{-\lambda \pi r^2 s}$ in the case 1. In order to study the asymptotically stability of the replicator dynamics (15) around the unique ESS, we linearize (15) at $s^* = s^*_1.$ We obtain the following linear delay differential equation

$$\dot{x} = -\mu s^*(1-s^*)\alpha \left( 1 + s^*(1-s^*)\lambda \pi r^2 \right)$$

$$\times \left( ((V + \Delta)x(t - \tau_T) + \kappa x(t - \tau_S)) \right)$$

where $x(t) = s(t) - s^*.$

The following theorem give sufficient conditions of stability of (14) at zero.

**Theorem 6 (see [10]):** Suppose at least one of the following conditions holds

- $(V + \Delta)\tau_T + \kappa \tau_S < \frac{1}{s^*(1-s^*)\mu \alpha (1+s^*(1-s^*)\lambda \pi r^2)}$
- $(V + \Delta)\tau_T < \frac{V + \Delta - \kappa}{s^*(1-s^*)\mu \alpha (1+s^*(1-s^*)\lambda \pi r^2)}$
- $(V + \Delta)\tau_T < \frac{-V - \Delta + \kappa}{s^*(1-s^*)\mu \alpha (1+s^*(1-s^*)\lambda \pi r^2)}$

Then the ESS $s^*$ is asymptotically stable.

A necessary and sufficient condition of stability of (14) at zero when delays are symmetric is given in theorem 7.

**Theorem 7 (symmetric delay):** Suppose that $\tau_T = \tau_S = \tau.$ Then, the ESS $s$ is asymptotically stable if and only if

$$\tau < \frac{\pi}{2s^*(1-s^*)\mu \alpha (1+s^*(1-s^*)\lambda \pi r^2)(V + \Delta + \kappa)}$$

D. Numerical investigation

We describe numerical application of our evolutionary game model with the dirac and Poisson distribution of nodes under the delayed replication dynamics.

1) Dirac distribution: The numerical solutions given figure 16 are obtained when the random variable $K$ is Dirac $\delta_{\{n\}}.$ We took $n = 4 = k_{\max}, \Delta = \frac{1}{4} = \delta, V = 1.$ The initial condition is 0.02 and the delays $\tau_T$ and $\tau_S$ between 0.02 and 7. For the small delays: $\tau_T = 0.02, \tau_S = 0.02$ and $\tau_T = 3, \tau_S = 2$ respectively, the system is stable. For the delay $\tau_H = 7$ and $\tau_S = 5,$ the system is unstable and the proportion of transmitters in the cell oscillates around the ESS.
2) Poisson distribution: We took $n = 4 = k_{\text{max}}$, $\Delta = \frac{1}{4} = \delta = \kappa$ and $V = 1$. The initial condition in all these figures is 0.02.

The delays $\tau_T$ and $\tau_S$ are between 0.02 and 7. The fraction of transmitters in the population is represented in figure 17 for $\lambda = 0.5$ and $r = 1$.

In the figures 18,19 and 20 we compare evolution of the fraction of transmitters varying the parameter of density $\lambda$ between 0.1 and 5 for the case 1, 2 and 3 respectively. In all these figures, the time delays are respectively 3 and 2. Note that in all these figures the equilibrium point is decreasing function in the density parameter $\lambda$.

Figure 21 compare the three cases for $\lambda = 0.5$. The case 1 prime(resp. 2 prime) represents the proportion of transmitters with at least one opponent in the case 2 (resp. case 2).

VI. CONCLUDING REMARKS

In this paper, we considered a wireless network consisting of population of nodes. Our achievements are of evolutionary stable strategy and evolution of the system. We showed that the evolutionary symmetric MAG has an unique ESS under dirac distribution and poisson distribution.

In perspective we want to generalize the model in order to take into consideration some mobility models for the node distribution. We are interesting also in some power control algorithms that can be obtained in such wireless network environment and study the link of our model with cooperation principles in ad hoc networks.

REFERENCES


Fig. 18. Evolution of the fraction of transmitters varying the density parameter $\lambda$ in the case 1.

Fig. 19. Evolution of the fraction of transmitters varying the density parameter $\lambda$ in the case 2.

Fig. 20. Evolution of the fraction of transmitters varying the density parameter $\lambda$ in the case 3.

Fig. 21. Evolution of the fraction of transmitters for $\lambda = 0.5$ in poisson distribution. The time delays are respectively 3 and 2.