Stochastic modeling and analysis of adaptive voter models

Presentation by Vineeth S. Varma
Based on work with Emmanuel Kravitzch, Yezekael Hayel and Antoine Berthet

Research Center for Automatic Control (CRAN), CNRS,
Université de Lorraine, Nancy, France.
vineeth.satheeskumar-varma@univ-lorraine.fr

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Section 1

Research context & motivation

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Only one social interaction is modeled: mimetism.

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Definition 1.1 (Homophily)

Homophily is the natural trend one has to connect with alike people.

Definition 1.2 (selective exposure)

dismiss dissonant information and gainsayers.

Section 2

The model

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The Adaptive Voter Model (AVM)

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- two additional edge-dynamics that are **Node-centric**:
  1. a **link-breaking** procedure; agent l picks u.r an $m \in N_l$ and breaks his directed link *only if* $x_l(t^-) \neq x_m(t^-)$: selective exposure
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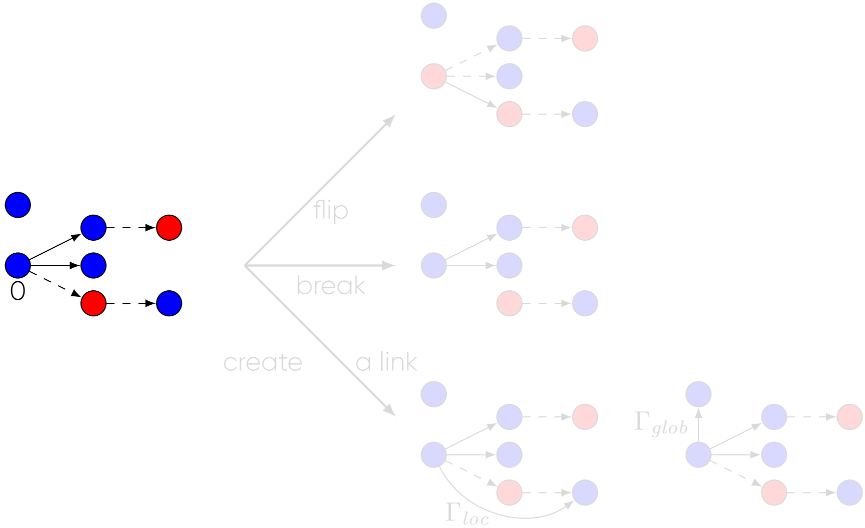
Or **Edge-centric** rules: i.e., each edge has an independent link breaking/creation rate.

An example in picture

Let us focus on agent 0:



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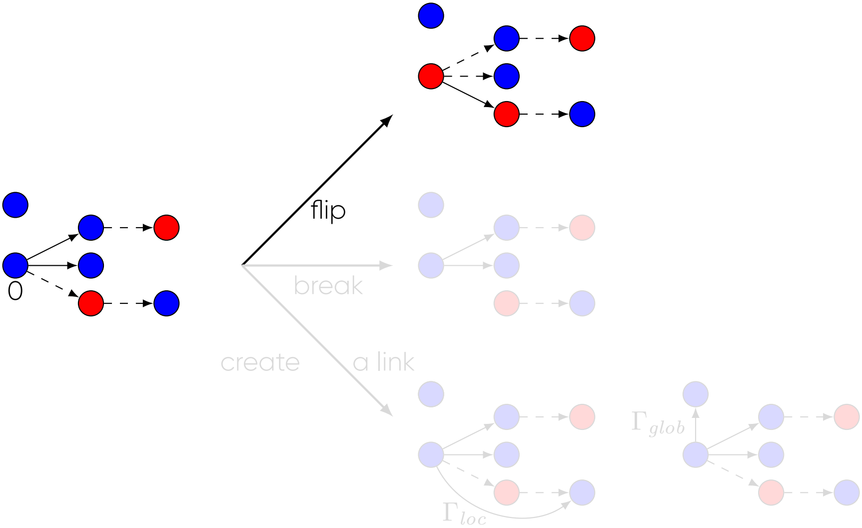

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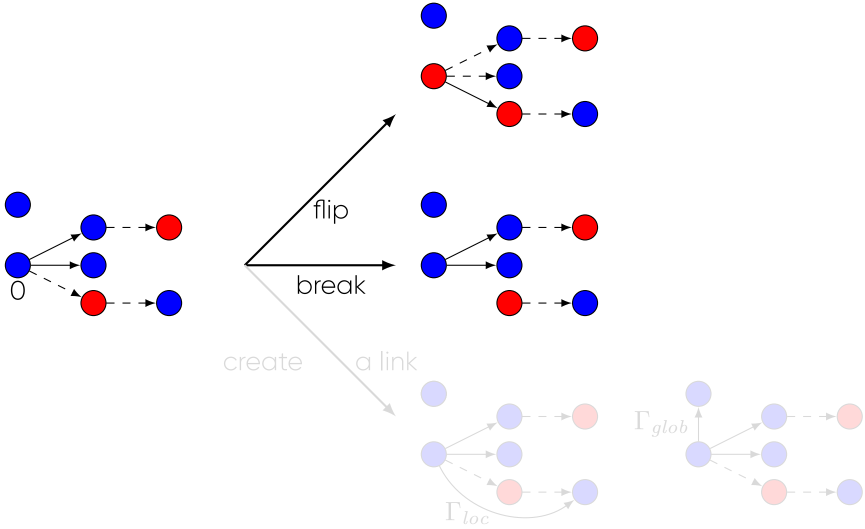

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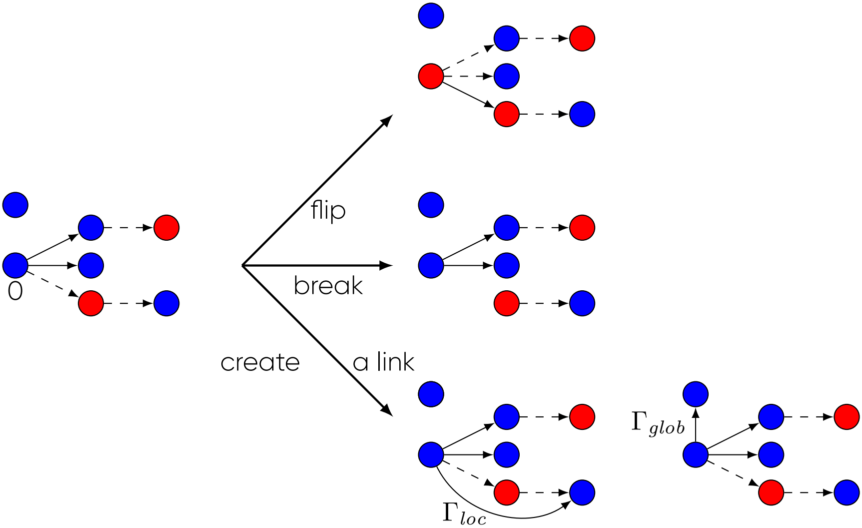

Related literature

The classical VM has been extensively analyzed with various refinements:

- Originally analyzed on the infinite lattice $\mathbb{Z}^d$ in [Lig12], and on a tree-like interaction network [Lig+99];
- based on a group-pressure mechanism[1] [CMP09; Mob15];
- under heterogeneous networks [SAR08];
- using the majority rule [Yil+10].

The AVM (Adaptive VM) also attracts a growing attention:

- most of the work focuses on global linkage [Dur+12; GB08];
- the 2-hop linkage [2] is also studied but in a lesser extent [Mal+16; RMS18];

Mostly, there always are only two parameters: the flip intensity ϕ and the ratio $\frac{\gamma}{\beta}$: nodes break and rewire instantaneously.

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Node-centric systems

Normalizing with the degree, we get a more agent-based system:

- Local linkage:
  $\overline{\Gamma}_{glob}(lm; x, A) = \gamma(1 - a_{lm}) \sum_j \frac{a_{lj}}{\text{deg}(l;A)} \frac{a_{jm}}{\text{deg}(j;A)} \underbrace{\mathbb{1}_{(x_l = x_m)}}_{\text{Homophily}}$:
  agent l picks a neighbour j U.R among N_l and then picks an agent UR among N_j, and gets eventually connected if they have same spin.
- flip: $\overline{\Phi}(k; x, A) = \phi \sum_j \frac{a_{kj}}{\text{deg}(k;A)}$: agent k picks UR one of its neighbour and copies its spin.
- Break: $\overline{B}(lm; x, A) = \beta \frac{a_{lm}}{\text{deg}(l;A)} \underbrace{\mathbb{1}_{(x_l \neq x_m)}}_{\text{Sel. Exp.}}$

Limit points of a Markov Process

Definition 3.1 (absorbing points)

An absorbing point ∂ of a Markov process $(Z_t)_t$ is a state such that

$$Z_{t_0} = \partial \implies Z_t = \partial \ \forall t \geq t_0. \qquad (1)$$

Remark 3.2

There can be several absorbing points, but it is still stronger than an absorbing set: when Z reaches a ∂, then it stays ***constant*** *forever.*

For the AVMs under consideration, the absorbing points are attractive in the following sense:

$$T_{abs} := \inf \left\{ t > 0 : (X^K, A^K)(t) \text{ reaches an absorbing point} \right\}.$$

In addition, T_{abs} is of finite mean: $\mathbb{E}\, T_{abs} < \infty$ (and then a.s finite).

The absorbing configurations for Γ_{glob}

If we take global linking Γ_{glob}, the **absorbing points** are:

$$\mathcal{A}_{glob} = \Big\{(x, a) \in \mathcal{S}_K : \forall (l, m) \in [K]^2, (x_l = x_m \text{ and } a_{lm} = 1) \text{ or } (x_l \neq x_m \text{ and } a_{lm} = 0)\Big\}$$

An absorbing state is then a clustered configuration where

- $C^+ \bigcup C^- = [K], C^+ \bigcap C^- = \emptyset$,
- with $x_k = +1 \; \forall l \in C^+$ and $x_k = -1 \; \forall k \in C^-$,
- and with no links between the two blocks: $a_{C^+ C^-} = a_{C^- C^+} = 0$.

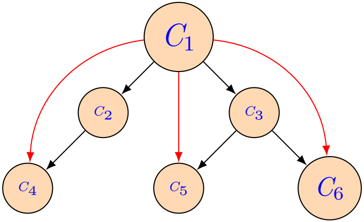
The absorbing configurations for Γ_{loc}

If we take local linkage Γ_{loc}, the **absorbing points** are:

$$\mathcal{A}_{loc} = \Big\{(x,a) \in \mathcal{S}_K : k \xrightarrow{\text{path}} j \implies a_{kj} = 1 \text{ and } x_k = x_j\Big\}.$$

Remark 3.3

*In the case of $\Gamma = \Gamma_{loc}$, there can be much more than two clusters because when two sets $U, V \subset [K]$ get disconnected: $a_{lm} = 0$ $\forall (l,m) \in U \times V$, then they stay disconnected **forever**.*



1-path-length completion:

$$\forall k \in C_1, j \in C_6, a_{kj} = 1.$$

C_p are all complete graphs:
$\forall i, j \in C_p, a_{ij} = 1.$

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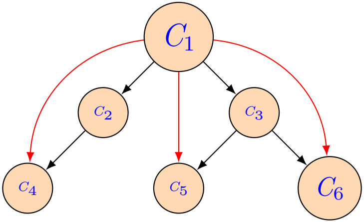

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How does discordance survive ?

Definition 3.4 (Discordance)

We say an edge lm is discordant if $a_{lm}\mathbb{1}_{(x_l \neq x_m)} > 0$. We can define total discordance of any configuration $(x, a) \in \mathcal{S}_K$ as

$$\mathcal{D}(x, a) := \sum_{lm} a_{lm}\mathbb{1}_{(x_l \neq x_m)} \quad (2)$$

Definition 3.5

slow extinction Define $T_K := \inf\{t > 0 : \mathcal{D}(X(t), A(t)) = 0\}$. We say that discordance *slowly extincts* if

$$\exists c > 0, \mathbb{P}\left(T_K < e^{cK}\right) < e^{-cK}. \quad (3)$$

For which values of ϕ, β, γ does the discordance slowly extinct ?

Section 4

A discrete time free global-linkage AVM

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Model

We have a discrete time Markov process described by $(X(t), A(t))$, described as follows. For all $i \in \mathcal{V}$ and $t \in \{1, 2, \dots\}$, we have

$$\Pr(X_i(t+1) = -X_i(t)) = \frac{\phi}{N} \frac{N_i^-(t)}{\max\{1, N_i(t)\}}. \qquad (4)$$

Additionally, for any $j \in \mathcal{V}$,

$$\Pr(A_{i,j}(t+1) = 0 | A_{i,j}(t) = 1) = \beta \frac{1}{\max\{1, N_i^-(t)\}} \qquad (5)$$

and

$$\Pr(A_{i,j}(t+1) = 1 | A_{i,j}(t) = 0) = \gamma \frac{1}{\max\{1, N - N_i(t)\}} \qquad (6)$$

where $N_i(t) = \sum_{j=1}^{N} A_{i,j}(t)$ and $N_i^-(t) = \sum_{j=1}^{N} A_{i,j}(t) 0.5 |X_i(t) - X_j(t)|$.

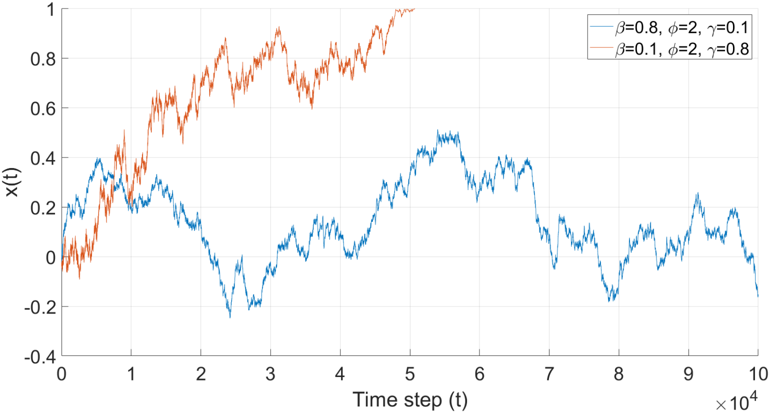
Metastability criteria (conjecture)

Define: $x(t) := \frac{\sum_{i=1}^{N} X_i(t)}{N}$,
$N_S(t) = \sum_{i=1}^{N} \max 0, SX_i(t)$,
$a_{S_1 S_2} = \frac{\sum_{i,j}^{N} a_{i,j} 1(X_i = S_1)(X_j(t) = S_2)}{\max\{1, N_{S_1}(t)\} \max\{1, N_{S_2}(t)\}}$.

Approximation and metastability

Fr $\beta >> \gamma$, $a_{+1,-1} > a_{-1,+1}$ (eventually) when $x < 0$
This leads to $x = 0$ being an attractive.

Simulations



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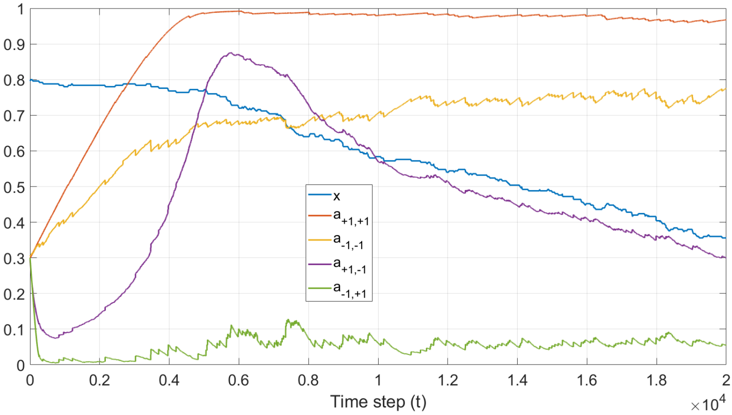

Illustrative simulation with $\phi = 0.1$

Section 5

Local linkage: study of an interesting particular case

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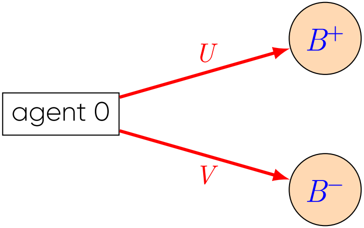
Description of the initial configuration

A unique agent labeled agent 0 is under the influence of two cliques B^+ and B^- of large size with opposite orientation:



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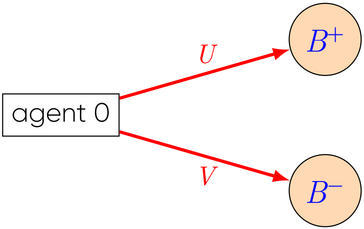
- $x_j = \sigma 1$ for $j \in B^\sigma$;
- the two blocks are of **same size**: $|B^+| = |B^-| = K >> 1$
- the two blocks are **static complete graphs**: $\forall t \geq 0, a_{lm}(t) = 1$ for all $(l, m) \in (B^+)^2 \bigcup (B^-)^2$.
- Furthermore, the two blocks stay **totally disconnected one with the other**: $a_{ml} = a_{lm} = 0 \; \forall (l, m) \in B^+ \times B^-$.



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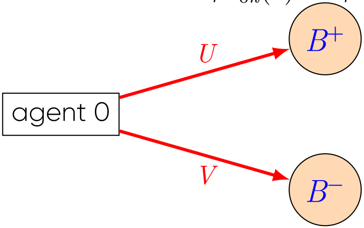
- Furthermore, the two blocks stay **totally disconnected one with the other**: $a_{ml} = a_{lm} = 0 \ \forall (l, m) \in B^+ \times B^-$.
- At initial time, $a_{0k}(0) = 1$, for all $k \in B^+ \bigcup B^-$.



$$U(t) := \sum_{k \in B^+} a_{0k}(t) \text{ and } \overline{U}(t) := \frac{1}{K} U(Kt)$$

$$V(t) := \sum_{k \in B^-} a_{0k}(t) \text{ and } \overline{V}(t) := \frac{1}{K} V(Kt).$$

the ODE approximation

Proposition 5.1 (long-term behaviour)

Let T *be a finite horizon time. the system* $\left(\bar{U}, \bar{V}\right)$ *can be approximated as follows:*

$$d\bar{U}(t) = F_1(\bar{U}, \bar{V})dt + d\epsilon_u(t), \quad (7)$$

$$d\bar{V}(t) = F_2(\bar{U}, \bar{V})dt + d\epsilon_v(t), \quad (8)$$

with $F = (F_1, F_1) : [0,1]^2 \longrightarrow [0,1]^2$ *being the following vector field:*

$$F_1(u, v) = \frac{u}{(u+v)^2}\left(\gamma(1-u)u\mathbb{1}_{(u<1)} - \beta v\mathbb{1}_{(u>0)}\right), \quad (9)$$

$$F_2(u, v) = \frac{v}{(u+v)^2}\left(\gamma(1-v)v\mathbb{1}_{(v<1)} - \beta u\mathbb{1}_{(v>0)}\right), \quad (10)$$

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two regimes

Proposition 5.2

For $\gamma > \beta$, a first equilibrium $p = (w^*, w^*)$ appears on the diagonal, with $w^* = 1 - \frac{\beta}{\gamma}$. Moreover, for $\gamma > 3\beta$, two extra (unstable) equilibria q_1, q_2 appear and p becomes stable.

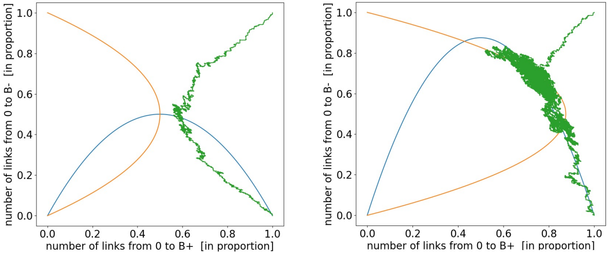

Figure 1: The green curve corresponds to the trajectory of $(\overline{U}, \overline{V}) \in [0,1]^2$.
For $\frac{\gamma}{\beta} = 3.1$, persistent hesitation occurs (right).
On the contrary, agent 0 is quickly convinced when $\frac{\gamma}{\beta} = 2.9$ (left).

Open questions

- What kind of additional results can be obtained for the general local linkage $\Gamma = \Gamma_{loc}$?
- Does a limiting (deterministic) system exist ?

Section 6

Ongoing work

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Stochastic rates: edge-centric

- Free-global linkage: $\Gamma_{fglo}(lm; x, A) = \gamma(1 - a_{lm})$,
  OR
- Global linkage: $\Gamma_{glob}(lm; x, A) = \gamma(1 - a_{lm}) \underbrace{\mathbb{1}_{(x_l = x_m)}}_{\text{Homophily}}$,

  OR
- Local linkage: $\Gamma_{loc}(lm; x, A) := \gamma(1 - a_{lm}) \sum_j a_{lj} a_{jm} \underbrace{\mathbb{1}_{(x_l = x_m)}}_{\text{Homophily}}$
- Break: $B(lm; x, A) = \beta a_{lm} \underbrace{\mathbb{1}_{(x_l \neq x_m)}}_{\text{Sel. Exp.}}$
- flip: $\Phi(k; x, A) = \phi \sum_j a_{kj} \mathbb{1}_{(x_l \neq x_m)}$ (standard voter model scheme)

Discordance

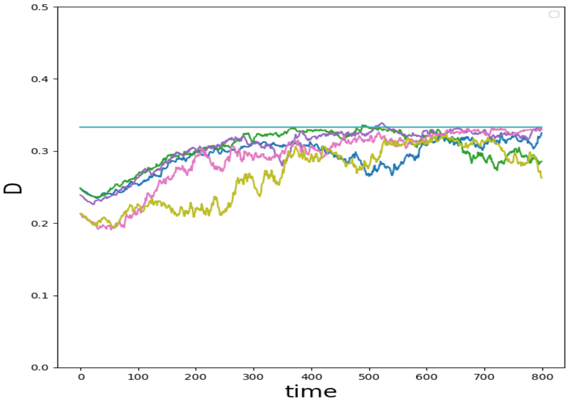
Definition 6.1 (discordance)

The total discordance $\mathcal{D}(x, a)$ of any configuration $(x, a) \in \mathcal{S}_K$ is defined as:

$$\mathcal{D}(x, a) := \frac{1}{K^2} \sum_{lm} a_{lm} \mathbb{1}_{(x_l \neq x_m)} = \frac{1}{K^2} \Big(a_{C^+ C^-} + a_{C^- C^+} \Big). \quad (11)$$

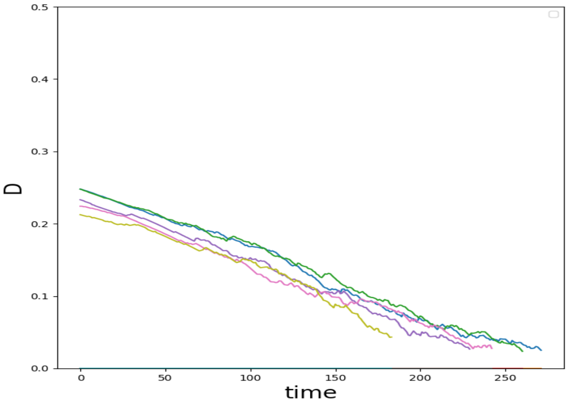
If $\mathcal{D}(x, a) = 0$, then we almost surely reach an AS (since the flip process stops).

Simulations



Five trajectories of the discordance with $K = 500$, $(\beta, \gamma) = (1, 4)$ and $\phi = 6$.

Simulations



Five trajectories of the discordance with $K = 500$, $(\beta, \gamma) = (1, 4)$ and $\phi = 1$.

Discussions and future works

1. External entities (advertisers, polical campaigns etc.)controlling the opinions or graph dynamics
2. Consider non-VM OD models: continous OD and a discrete graph dynamics: studied by Krause, Frasca etc..
   So why not a continuous evolving graph?
3. Application of similar models to other frameworks like epidemics.

Questions?

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Section 7

References

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