Computing the bias of mean field approximation

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joint work with Sebastian Allmeier (Inria) and Benny Van Houdt (Univ. Antwerp)

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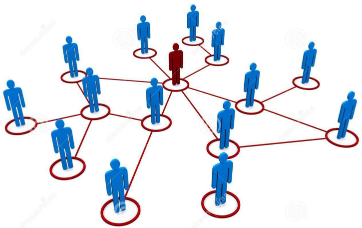
Stochastic information models on 'dense' graphs



- Propagation of an information over time
- Steady-state properties (e.g. % of informed people)

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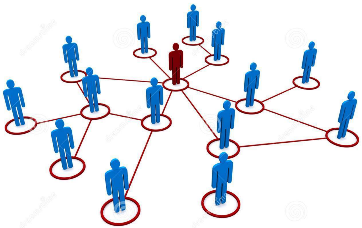

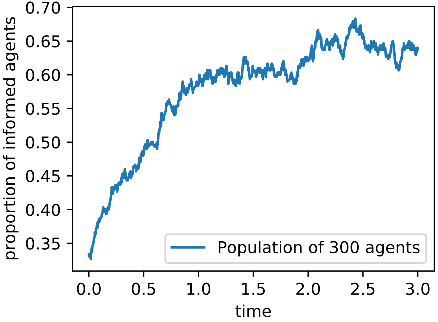

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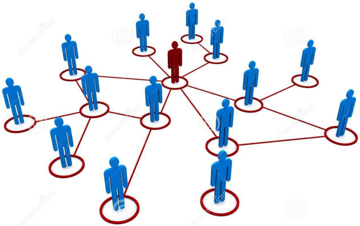

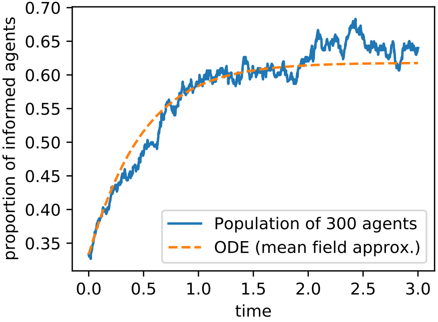

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Objective of the talk (and outline)

1. What is mean field approximation?
2. How to characterize the bias of this approximation?
3. What about multiscale models?
4. Conclusion

Outline

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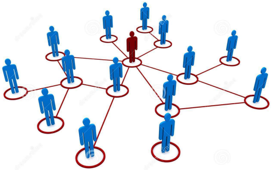
Running example: Simple information propagation model.



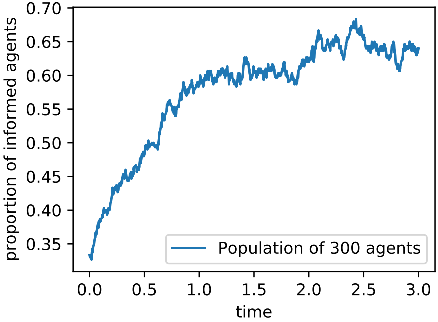
Population of n persons where each person can be "Informed" or "Outdated". x is the proportion of "informed" people.

- Informed persons loose information at rate 1.
- Outdated persons become informed at rate $1 + x$

Stochastic model

If X is the proportion of "informed" people, then:

$$X \mapsto X - \frac{1}{n} \text{ at rate } nX$$
$$X \mapsto X + \frac{1}{n} \text{ at rate } n(1-X^2) = n(1-X)(1+X)$$



Mean field approximation

$$X \mapsto X - \frac{1}{n} \text{ at rate } nX \qquad \textit{average change: } -X$$

$$X \mapsto X + \frac{1}{n} \text{ at rate } n(1 - X^2) \qquad \textit{average change: } 1 - X^2$$

The mean field approximation is the solution of the ODE $\dot{x} = 1 - x^2 - x$.

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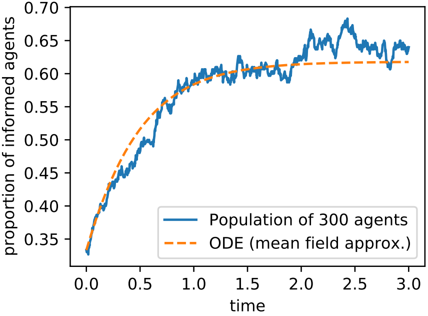

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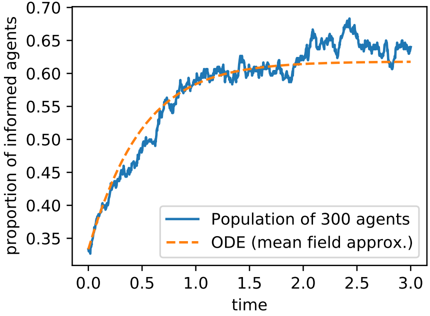

With this approximation, we study:

- The transient regime.
- The fixed point: $x(\infty) = (\sqrt{5} - 1)/2$.

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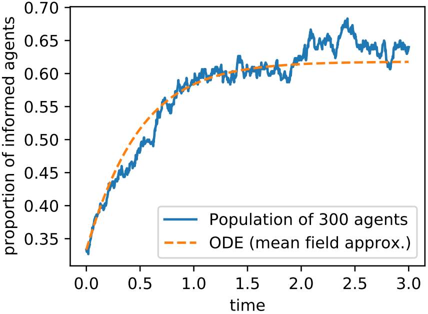

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How accurate is this approximation?

Accuracy of the mean field approximation

n	5	10	100	∞
$\mathbb{P}$(someone informed)	0.593	0.601	0.61679642	$(\sqrt{5}-1)/2 \approx 0.618$.
Error	0.025	0.125	0.0012	0

Table: Steady-state values

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Theorem: (G. Bortolussi, Tribastone) For this model, if $\mathbb{E}\left[X^{(n)}\right]$ is the probability that someone is "informed" and if the $x(\infty) = (\sqrt{5}-1)/2$ is the mean field approximation, then:

$$\mathbb{E}\left[X^{(n)}\right] = x(\infty) + \frac{1}{20n}(\sqrt{5}-1) + \frac{1}{50n^2}(\sqrt{5}-3) + O(1/n^3).$$

Outline

We study a generic interaction model

We consider a population of n objects with two types of interactions:

- Unilateral transitions:

  Object k jumps from state i to j at rate $r_{ij}^{(k)}$

- Pairwise interactions:

  Object k, k' *simultaneously* jump from states (i, i') to (j, j') at rate $r_{ij,i'j'}^{(k,k')}/n$

If the rates do not depend on k, we call the model homogeneous.

Mean field approximation for homogeneous models

$$X_s^{(n)}(t) = \frac{1}{n}\{\# \text{ objects in state } s \text{ at } t\}$$

The transitions are:

$$\mathbf{X}^{(n)} \to \mathbf{X}^{(n)} + \frac{1}{n}(e_j - e_i) \qquad \text{at rate } n r_{ij} X_i.$$

$$\mathbf{X}^{(n)} \to \mathbf{X}^{(n)} + \frac{1}{n}(e_j - e_i + e_{j'} - e_{i'}) \qquad \text{at rate } n r_{ij,i'j'} X_i X_{i'}.$$

This is a density dependent population process (Kurtz 70s).

(one example is our information propagation model)

Mean field method for non-homogeneous models

$X_s^{(n)}(t) = \frac{1}{n}\{\#\text{ objects in state } s \text{ at } t\}$ $\Rightarrow X^{(n)}$ is not Markovian.

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Solution: represent model using indicators:

$$Y_{(k,s)}^{(n)}(t) = \begin{cases} 1 & \text{if object } k \text{ is in state } s \text{ at time } t \\ 0 & \text{otherwise} \end{cases}$$

$\mathbf{Y}^{(n)}$ is Markovian.

$$\mathbf{Y}^{(n)} \to \mathbf{Y}^{(n)} + e_{k,j} - e_{k,i} \qquad \text{at rate } r_{ij}^{(k)} Y_{k,i}.$$

$$\mathbf{Y}^{(n)} \to \mathbf{Y}^{(n)} + e_{k,j} - e_{k,i} + e_{k',j'} - e_{k',i'} \qquad \text{at rate } r_{ij,i'j'}^{(k,k')} Y_{k,i} Y_{k',i'}.$$

Mean field approximaiton and result

The drift is:

$$f(\mathbf{y}) = \sum_{\text{all transitions}} \text{Transition change for } \mathbf{y} \times \text{Rate of transition at } \mathbf{y}.$$

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Theorem (Allmeier, G. 2022) There exists an (easily computable) vector $V(t)$ such that for any finite time:

$$\mathbf{P}\left[\text{ Object } k \text{ is in state } s \text{ at } t\right] = y_{k,s}(t) + \frac{1}{N} V_{k,s}(t) + O(\frac{1}{N^2}).$$

$V_{k,s}(t)$ is the bias of the mean field approximation.

Idea of proof

1. We can show that $\mathrm{cov}(Y_{k,s}Y_{k',s'})(t) = \frac{1}{N}W(t) + O(1/N^2)$, where $W(t)$ satisfies a (time inhomogeneous) linear ODE:

$$\dot{W} = A(y(t))W + WA^T(y(t)) + Q(x(t)).$$

2. We then have $\mathbb{E}\left[Y_{k,s}(t)\right] = y(t) + V(t)$, where $V(t)$ satisfies a (time inhomogeneous) linear ODE:

$$\dot{V} = A(y(t))V + B(x(t)) \cdot W(t).$$

Where does the $1/N$-term

The moment closure approach

Consider a system for which X becomes $X + 1/n$ at rate nX^2. We have:

$$\frac{d}{dt}\mathbb{E}\left[X\right] = \mathbb{E}\left[X^2\right]$$

This equation is not closed

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$$\frac{d}{dt}\mathbb{E}\left[X^2\right] = 2\mathbb{E}\left[X^3\right] + \frac{1}{n}\mathbb{E}\left[X^2\right]$$

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$$\frac{d}{dt}\mathbb{E}\left[X^2\right] = 2\mathbb{E}\left[X^3\right] + \frac{1}{n}\mathbb{E}\left[X^2\right] \qquad \approx 2(3\mathbb{E}\left[X^2\right]\mathbb{E}[X] - 2\mathbb{E}[X]^2) + \frac{1}{n}\mathbb{E}\left[X^2\right]$$

(refined approximation)

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$$\frac{d}{dt}\mathbb{E}\left[X^3\right] = \mathbb{E}\left[\frac{3X^4}{n} + \frac{4X^3}{n^2} + \frac{X^2}{n^3}\right]$$

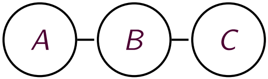
$$\vdots$$

The moment equations are never closed.

- They can be closed by assuming $\mathbb{E}\left[(X - \mathbb{E}[X])^d\right] \approx 0$
- This gives a $O(1/n^{\lfloor (d+1)/2 \rfloor})$-accurate approximation.

Outline

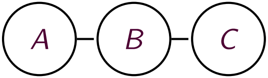
Communication with interference



Interference graph.

n nodes per class A, B or C

Communication with interference



Interference graph.

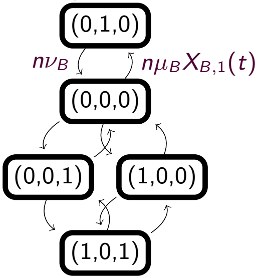
n nodes per class A, B or C

State is X, Y

- $X_{i,s} =$ proportion of nodes of class i with $\geq S$ messages.
- $Y_i = 1$ if class i talks.

This is a two timescale model

"Fast process": Y.



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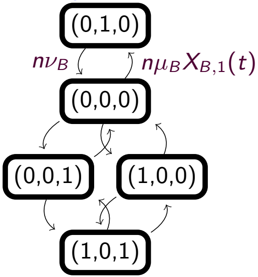

“Slow process”: X.

Arrival/departure:

$$X_{i,s} \mapsto X_{i,s} \pm \frac{1}{N}$$

Rate depends on y.

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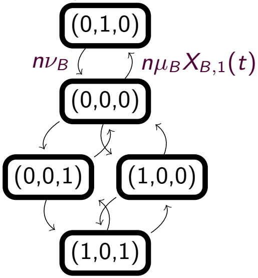

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Two approximations:

$\mathbf{P}[Y(t) = y] \approx \pi_y(X(t))$ | Drift $f(X, Y)$

$$\dot{x} = \sum_y \pi_y(x) f(x, y)$$

(Averaging technique):

Accuracy results (Allmeier, G. 2022)

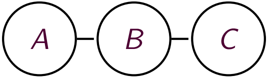
Theorem. If $X(t)$ is the two timescale process, if the rates are twice differentiable and the evolution the the fast process is "unichain", then:

$$\mathbb{E}[X(t)] = x(t) + \frac{1}{N}C(t) + O(1/N^2).$$

Holds uniformly in time if the ODE has an exponentially stable attractor.

Numerical example

With $n = 1$ node per class!



Jobs arrive at rate 1, activation rate $= 3$. Job duration is $1/3$.

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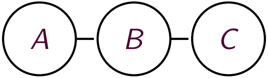

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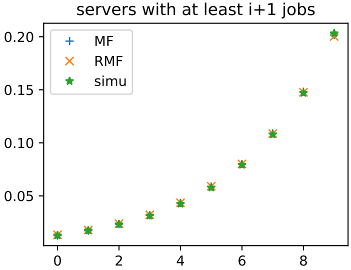
Class A or C



Class B

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Conclusion

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- Numerical library: `https://pypi.org/project/rmftool/`

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We question its validity / accuracy.

- We characterize the bias for different models (smooth homogeneous, heterogeneous, multi-scale).
- To do so, we take correlations into account.
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Many open questions: (sparse) geometric models, non-Markovian, controlled systems

More slides and references: `http://polaris.imag.fr/nicolas.gast`

References

Results on which this talk is based:

- Mean Field and Refined Mean Field Approximations for Heterogeneous Systems: It Works! by Allmeier and Gast. SIGMETRICS 2022.
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