Opinion Dynamics

Questions, Answers, and further Questions

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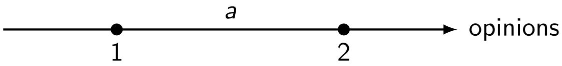
Contents

Simple Examples

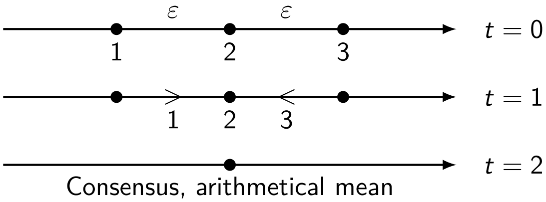
Examples of scalar opinions of n agents with confidence level ε.

$n = 2$



$\varepsilon < a$ dissent forever, $\varepsilon \geq a$ consensus in next period by arithmetical mean

$n = 3$



$n \leq 4, n \leq 5$ Similarly, consensus at $t = 5,\ t = 6$.

$n = 6$ What do you expect?

Simulation for six Agents

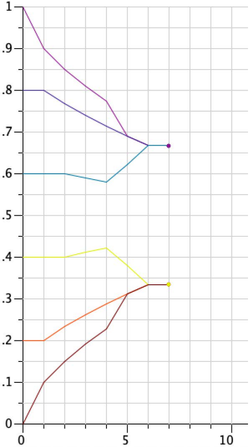


Figure: $n = 6$ agents, equally spaced with $\varepsilon = 0.2$, two clusters of size 3, freezing time $T = 6$.

First Results

Question: Any n, agents equally spaced by ε?

- Peter Hegarty and Edvin Wedin. "The Hegselmann-Krause dynamics for equally spaced agents". In: *Journal of Difference Equations and Applications* 22.11 (2016), pp. 1621–1645

Case: $n = 6k + \ell, \ \ 0 \leq \ell \leq 5,$ equally spaced by ε

- After every fifth time step 3 agents will be disconnected from either end and then collaps to a cluster in the next time step.
- After finite time the final confirmation consists of $2k$ clusters of size 3.
- Proof not trivial

Question: What if agents are not equally spaced?

- **Case:** $n \leq 4$: consensus $\Leftrightarrow$ max difference at $t = 0$ between neighbours $\leq \varepsilon$.
- This is not true for $n \geq 5$ (not obvious).

Contents

Bounded Confidence Model in Higher Dimensions

- **Agents** $1, 2, \dots, n$

Opinion Profile

- **Opinion profile** of agent i at time $t = 0, 1, 2, \dots$ given by $(x^1(t), x^2(t), \dots, x^n(t))$
- d-dimensional opinions $x^i(t) \in \mathbb{R}^d$ with d aspects

Confidence Set

- **Confidence set** of agent i for **confidence level** ε and profile $x = (x^1, x^2, \dots, x^n)$ given by

$$I(i, x) = \{1 \leq j \leq n \mid \|x^i - x^j\| \leq \varepsilon\}.$$

- $\|\cdot\|$ norm on $\mathbb{R}^d$.

Dynamics

- $$x^i(t+1) = \frac{1}{\#I(i, x(t))} \cdot \sum_{j \in I(i, x(t))} x^j(t)$$

  = arithmetical mean of all agents in the confidence set of agent i

Bounded Confidence Model in Higher Dimensions

Matrix Model

- $x(t+1) = A(x(t)) \cdot x(t) \quad \text{with} \quad a_{ij}(x) = \begin{cases} \frac{1}{\#I(i,x)}, & \text{for } j \in I(i,x) \\ 0, & \text{otherwise} \end{cases}$
- Dynamics state dependent on $x(t)$, nonlinear

More General Setting

- Multiagent system in higher dimensions
- $x(t+1) = A(t) \cdot x(t)$ with $x(0) \in (\mathbb{R}^d)^n,\ t = 0, 1, 2, \dots$
- $A(t)$ row-stochastic $n \times n$-matrix with $a_{ij}(t) \geq 0$ and $\sum_j^n a_{ij}(t) = 1$.
- Positive dynamical system in discrete time, time-variant

Main Questions

A. Convergence of consensus $x^* = \lim_{t\to\infty} x^i(t)$ for $i \in \mathbb{N}$?

B. Convergence at all — if yes, to what limit pattern?

Simulations for 1D and 2D — Consensus

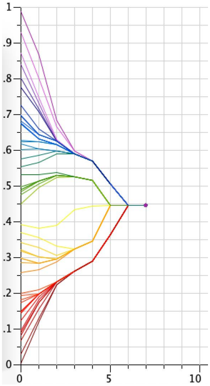


Figure: $n = 50$ agents in one dimension, start with a uniform random distribution, $\varepsilon = 0.25$, consensus at $T = 6$.

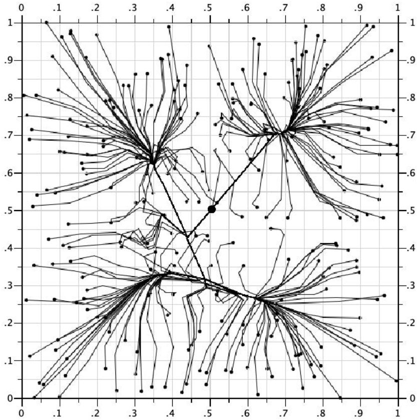


Figure: $n = 250$ agents in two dimensions, start with a uniform random distribution, $\varepsilon = 0.25$, consensus at $T = 21$.

Two Principles Considering Questions A and B

- Sequence of row-stochastic $n \times n$-matrices $A(t)$ for $t = 0, 1, 2, \ldots$

The structure of interaction is not too weak

- **Principle of the third agent**
- Define block $B := A(t) \cdot A(t-1) \cdot \cdots \cdot A(s)$ for $s \leq t$
- B is **scrambling** if for $i, j \in \mathbb{N}$ there exists a third agent k such that $b_{ik} > 0$ and $b_{jk} > 0$
- The sequence of $A(t)'$s is **blockwise scrambling** if there are infinitely many scrambling blocks

The intensity of interaction is not too small

- Define intensity: $\alpha(B) =$ (smallest positive entry of block B).
- **Block intensity rule**: $\sum_{B \in \varphi_s} \alpha(B)$ unbounded for a sequence φ_s of blocks

Example

Markov chain

$$x(t) = A^t x(0) \quad \text{for } t = 0, 1, 2, \ldots .$$

- Special case $A(t) = A$ for all t.
- Choose equal blocks $B = A^r$ for some $r \geq 1$.
- Block intensity $\alpha(B) > 0$
- Sequence of $A(t)'$s is blockwise scrambling if power A^r is scrambling

Proposition

Convergence to consensus for each $x(0)$ $\Leftrightarrow$ A power of A is scrambling

- Both principles are true

Contents

Theorem A

Theorem (Theorem A)

(i) *Any multiagent system in any dimension converges to a consensus* x^* *if the sequence of* $A(t)'s$

- *is blockwise scrambling*
- *the block intensity rules applies*

(ii) *If in addition*

- *length's of blocks are bounded by some* m
- $\min^+ A(t) \geq \alpha > 0$ *for all* t
- *sequence of blocks covers* $\mathbb{N}$

then convergence to x^* *is exponentially fast:*

$$\|x^* - x^i(t)\| \leq c(m, \alpha, t) \max_{j,k \in \mathbb{N}} \|x^j(0) - x^k(0)\| \quad \forall t \in \mathbb{N}_0, \forall i \in \mathbb{N},$$

where $c(m, \alpha, t) = (1 - \alpha^m)^{t/m - 1}$.

Theorem A

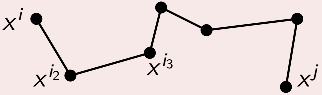
Corollary

*For opinion dynamics under bounded confidence ($\varepsilon > 0, x(0)$ given), the system converges to consensus exponentially fast with $c(n-1, 1/n, t)$ and in **finite time** if $x(t)$ is an ε-net for all t.*

Definition (ε-net)

A sequence $(x^1, x^2, \ldots, x^n)$ for $x^i \in \mathbb{R}^d$ is called an ε-net if for any $i, j \in \mathbb{N}$, there exist $i_1, i_2, \ldots, i_k \in \mathbb{N}$ with $i = i_1$ and $j = i_k$, such that

$$\|x^{i_1} - x^{i_2}\| \leq \varepsilon, \ldots, \|x^{i_{k-1}} - x^{i_k}\| \leq \varepsilon.$$



- In case of $d = 1$, $x_1 \leq x_2 \leq \cdots \leq x_n$: $\max_{1 \leq i \leq n-1} \|x_{i+1} - x_i\| \leq \varepsilon$.

Idea of the proof for Theorem A

Tools

- Row-stochastic $n \times n$-matrix A
- **Ergodic coefficient** $c(A) := 1 - \min_{i,j} \sum_{k=1}^{n} \min\{a_{ik}, a_{jk}\}$.

Steps

- $c(A)$ is the smallest $c \in [0,1]$, such that

$$\Delta(Ax) \leq c \cdot \Delta x \quad \text{for all } x = (x^1, \ldots, x^n) \text{ with } x^i \in \mathbb{R}^d.$$
$$\left(\Delta x = \max_{i,j \in \mathbb{N}} \|x^i - x^j\|\right)$$

- A scrambling $\quad \Leftrightarrow \quad c(A) < 1 \quad \Leftrightarrow \quad c(A) \leq 1 - \min^+ A$.
- Then $\Delta x(t) = \exp\left(-\sum_k \min^+ B_k(t)\right) \Delta x(0)$
- Block intensity rule $\quad \Rightarrow \quad \lim_{t \to \infty} \Delta x(t) = 0$.

Idea of the proof for the Corollary

- $A(t)$ given by $a_{ij}(t) = \begin{cases} (\#I(i,x(t)))^{-1}, & \text{if } j \in I(i,x(t)) \\ 0, & \text{otherwise} \end{cases}$
  $\Rightarrow A(t)$ has positive diagonal with $\min^+ A(t) \geq 1/n$
- $x(t)$ ε-net (for all t) $\Rightarrow$ Product of $(n-1)$ $A(t)'$s scrambling
- Finite time: $\Delta x(t) \to 0$ for $t \to \infty$.
  $\Rightarrow$ after some t_0 each agent has all others as neighbours
  $\Rightarrow$ $x(t)$ stays constant for $t \geq t_0 + 1$

Simulations for 1D and 2D — Fragmentation

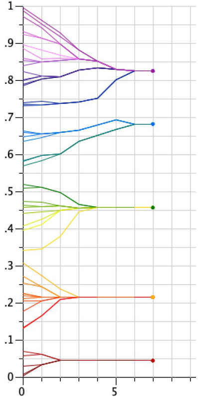


Figure: $n = 50$ agents in one dimension, start with a uniform random distribution, $\varepsilon = 0.1$, fragmentation at $T = 6$.

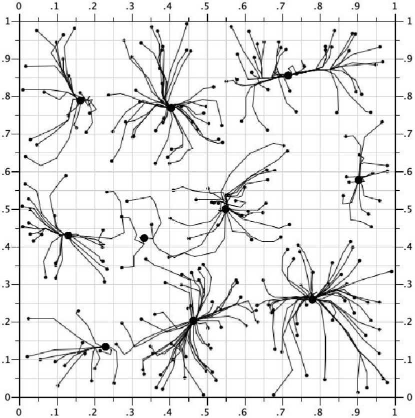


Figure: $n = 250$ agents in two dimensions, start with a uniform random distribution, $\varepsilon = 0.15$, fragmentation at $T = 7$.

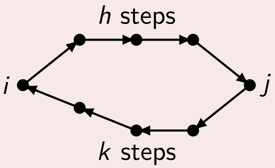
Reciprocal Interation

- Splitting can be achieved by reciprocal interaction.

Definition

A row-stochastic $n \times n$-matrix, where $a_{ij}^{(h)}$ is the the entry of A^h, is **reciprocal** if it has a positive diagonal and for all $i, j \in \mathbb{N}$, it holds that

$$a_{ij}^{(h)} > 0 \text{ for some } h \quad \Rightarrow \quad a_{ji}^{(k)} > 0 \text{ for some } k.$$



- The multiagent system is reciprocal if $A(t)$ is a reciprocal matrix for each t.

Theorem B

Theorem (Theorem B)

Let a multiagent system be reciprocal and $\min^{+} A(t) \geq \alpha > 0$ *for all* t.
Then fragmentation of opinions holds in any dimension in the sense that

(i) N *is the disjoint union of groups* $N(k)$ *with* $1 \leq k \leq m$ *and there exists* t_0, *such that for agents* i *and* j *in different groups it holds that* $a_{ij}(t) = 0$ *for* $t \geq t_0$.

(ii) $\lim_{t \to \infty} x^i(t) = c_k$ *for all* $i \in N(k)$ *for each* k.

Corollary

Theorem B holds for any system of opinion dynamics under bounded confidence.
The process becomes stationary in ***finite time***.

Idea of the proof for Theorem B

- Equivalence relation "$\sim$" in $\mathbb{N}$: For $i, j \in \mathbb{N}$, we define
  - $i \to j$: For each t there exists $t' \geq t$ with $a_{ij}(t') > 0$
  - $i \twoheadrightarrow j$: There exist $i_1, \dots i_r \in \mathbb{N}$ with $i \to i_1 \to \cdots \to i_r \to j$
  - $i \sim j$: $i \twoheadrightarrow j$ and $j \twoheadrightarrow i$
- "$\sim$" is reflexive, transitive and symmetric
- P equivalence class of $p \in \mathbb{N}$
- There exists a splitting time t_0 such that

  $$i \in P,\ j \notin P \quad \Rightarrow \quad a_{ij}(t) = 0 \text{ for } t \geq t_0$$

  $\Rightarrow A(t)$ can be restricted to P for $t \geq t_0$ and there blockwise scrambling.
- $A(t)$ reciprocal needed, a.o., for the block intensity rule.

Literature for both Theorems and the Corollaries

Based on joint work with Rainer Hegselmann since 2002

- Ulrich Krause. *Positive Dynamical Systems in Discrete Time - Theory, Models and Applications*. Berlin, München, Boston: De Gruyter, 2015. ISBN: 9783110365696. DOI: 10.1515/9783110365696
  Chapter 8 on "Dynamics of interaction: Opinions, mean maps, multi-agent coordination and swarms".
- Rainer Hegselmann and Ulrich Krause. "Consensus and Fragmentation of Opinions with a Focus on Bounded Confidence". In: *The American Mathematical Monthly* 126.8 (2019), pp. 700–716. DOI: 10.1080/00029890.2019.1626685
- Julien M. Hendrickx and John N. Tsitsiklis. "A new condition for convergence in continuous-time consensus seeking systems". In: *2011 50th IEEE Conference on Decision and Control and European Control Conference*. 2011, pp. 5070–5075. DOI: 10.1109/CDC.2011.6160231

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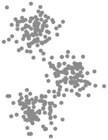
Microscope and Macroscope

Point Clouds



- How to aggregate or bundling data in a consistent manner? No law of distribution known.
- Taking a mean-arithmetic or geometric or. . . ?
- Local means versus global mean?

Microscope

- Successive discrimination of data by increasing **re-solution**

Macrosope

- Successive bundling of data by increasing **con-solution**

Construction of a Macroscope

- Bundling of data by using opinion dynamics under bounded confidence
- **Data collection** $x = (x^1, x^2, \dots, x^n)$ with $x^i \in D$ space of data in $\mathbb{R}^d$
- $z \in \mathbb{R}_+$ **scale of consolution** / confidence level

From the Corollary to Theorem B

- $f : D^n \times \mathbb{R}_+ \to D^n$ with $y = f(x, \varepsilon)$ defined by $y^i = x^i(T)$ for the process $x(t)$ given by $x(0), \varepsilon$ and becoming stationary in finite time T.
- **Macroscope** $M : D^n \to D^n$ for data space D defined by $M(x) = f(x, m(x))$.
- **Scale** $\varepsilon = m(x) = \min\{\|x^i - x^j\| : i, j \in \mathbb{N} \text{ with } x^i \neq x^j\}$
- Scale changes with x

Properties of the Macroscope

Increasing Consolution

- $m(M(x)) > 0 \quad \Rightarrow \quad m(x) < m(M(x))$

Robustness

- $m(x) \leq \varepsilon < m(M(x)) \quad \Rightarrow \quad f(\varepsilon, M(x)) = M(x)$

Macroscope Algorithm MAC

$M : D^n \to D^n$ with $M(x) = f(x, m(x))$ can be iterated

- $x \in D^n$ single data — nothing to bundle `stop`
  `otherwise` $m(x) > 0$, $f(x, \varepsilon) = x$ for $0 \leq \varepsilon < m(x)$
  `go to` $M(x) = f(x, m(x))$
- $M(x)$ single data — `stop`
  `otherwise` $m(M(x)) > m(x)$, $f(M(x), \varepsilon) = M(x)$ for $m(x) \leq \varepsilon < m(M(x))$
  `go to` $M^2(x) = M(M(x))$, etc.

Question

- Implementation on a computer?

Example of MAC

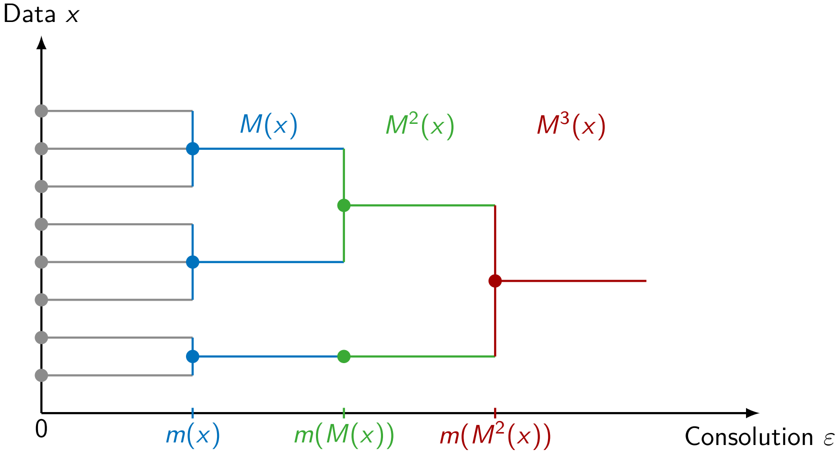


Figure: Dendrogram

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Application to Swarm Dynamics

- Bernard Chazelle. "The Total s-Energy of a Multiagent System". In: *SIAM Journal on Control and Optimization* 49.4 (2011), pp. 1680–1706. DOI: 10.1137/100791671
- Ulrich Krause. *Positive Dynamical Systems in Discrete Time - Theory, Models and Applications*. Berlin, München, Boston: De Gruyter, 2015. ISBN: 9783110365696. DOI: 10.1515/9783110365696
  Chapter 8.6
- Eitan Tadmor. *Emergent Behavior in Collective Dynamics*. Gibbs Lecture of the AMS. Apr. 2022. URL: https://www.youtube.com/watch?v=AenZz6Ooj2g
  Cucker-Smale Model of Swarm Dynamics

Contents

Agent Specific Confidence Levels

- Opinion dynamics under bounded confidence for $d = 1$
- Confidence set $I(i, x) = \{1 \leq j \leq n : \|x_i - x_j\| \leq \varepsilon_i\}$ for $1 \leq i \leq n$.
- Level of confidence dependent on agent i.

Question

- What can be said about consensus (Theorem A) and fragmentation (Theorem B)?

Agent Specific Confidence Levels

Literature

- Jan Lorenz. “Heterogeneous bounds of confidence: Meet, discuss and find consensus!” In: *Complexity* 15.4 (2010), pp. 43–52
  - Treats by simulation two different levels $\varepsilon < \varepsilon'$.
  - Surprising result: Consens neither for ε nor for ε' in the homogeneous case but consensus if some agents use ε, the other ones ε'.
- Anahita Mirtabatabaei and Francesco Bullo. “Opinion Dynamics in Heterogeneous Networks: Convergence Conjectures and Theorems”. In: *SIAM Journal on Control and Optimization* 50.5 (2012), pp. 2763–2785
  - A detailed mathematical analysis for ε_i for $1 \leq i \leq n$.
  - Main conjecture: Trajectory of each agent converges in finite time. Still an **open problem**, however:
    - Numerical evidence by simulations
    - Sufficient conditions are derived for the conjecture to hold true

Contents

Dependence of Consensus on the Confidence

- Ongoing work with Rainer Hegselmann and Malte Sieveking
- Opinion dynamics under bounded confidence, one dimension
- $x(0)$ and ε (uniform) given, $x(0)$ fixed in the following
- Theorem A on consensus, Theorem B on fragmentation

Question

- Can a change of ε lead from one to the other, especially

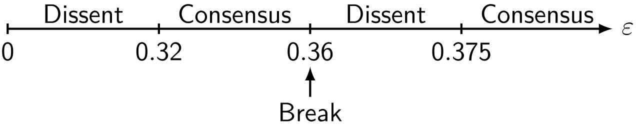
$$\text{Consensus for } \varepsilon \text{ and } \varepsilon < \varepsilon' \quad \overset{?}{\Longrightarrow} \quad \text{Consensus for } \varepsilon'$$

- Regarding the simple examples of the first Section: **NOT** for $n \leq 4$.

Dependence of Consensus on the Confidence

First Results

- R. Hegselmann found by computer simulations:
  - **YES** for $n \geq 5$
  - Consensus may break for higher confidence
  - Number of breaks increase with n
- M. Sieveking found analytically an example for $n = 5: \; x(0) \; = \; (0, 0.18, 0.36, 0.68, 1)$



- **Thus, more confidence may lead to dissent!**

Question

- How often does that happen — can breaks even accumulate?

Dependence of Consensus on the Confidence

- Call ε **robust** is there is an open neighbourhood $U(\varepsilon)$ of ε (in $[0, \infty)$), such that for $\varepsilon' \in U(\varepsilon)$ the processes for ε' and ε do coincide ($x(0)$ given).

Theorem

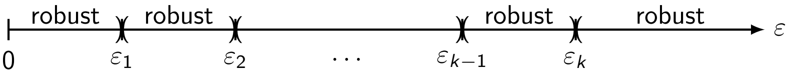
(i) *The set E of robust ε is open and dense in $[0, \infty)$.*

(ii) *The set of non-robust ε equals the boundary of E, which consists of isolated points.*

(iii) *The set of non-robust ε is finite.*

Dependence of Consensus on the Confidence

- In particular, there are only finitely many breaks and breaks cannot accumulate.



- $\varepsilon_1, \ldots, \varepsilon_k$ non-robust ε, especially breaks or changes from dissent to consensus.

Questions

- How does the number of non-robust ε (especially of breaks / phase transitions) depend on n?
- Is the Theorem true in higher dimensions?

Many thanks to Pascal Fernsel for his great help in preparing this presentation.

Thank you for your attention!