A hybrid model of opinion dynamics with memory-based connectivity

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Opinion dynamics as a networked system

Opinion dynamics

Opinion dynamics describes the **interaction between individuals exchanging opinions**.

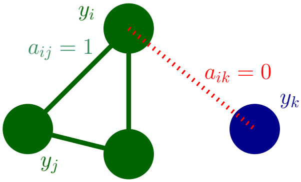
To predict the **evolution of social phenomena**.
To analyze **clustering patterns** and their **properties**.

The individuals interacting as a **networked system**:

- **Nodes** as **individuals** with **their own opinions** y_i.
- **Edges** a_{ij} as **exchanges of opinions**.

Opinion dynamics as a hybrid system [Frasca et al., 2019]

- **Networked system** as a **hybrid system**.
- **Opinions** ($y_i, i \in \{1, \dots, N\}$) evolving as a **continuous dynamics**.
- **Interactions** ($a_{ij},\ i, j \in \{1, \dots, N\}$) changing as a **discrete dynamics**.
- Hegselmann-Krause model, i.e. $|y_j - y_i| \leq C \Rightarrow$ interaction, i.e. $a_{ij} = 1$.
    - Convergence to clusters but instability.
    - Adaptive thresholds, i.e.$|y_j - y_i| \leq C(y, a) \Rightarrow$ interaction, i.e. $a_{ij} = 1$.
    - Attractivity and stability properties of the clustering pattern are guaranteed.



To take the past into account

Enemies

Buddies

- Exchange of options in social networks depends also on the interactions history.
- Hybrid model that:
  - Rules the interaction based on **local, instantaneous values** and the **past interactions**.
  - Exhibits a **clustering pattern** with **attractivity and stability properties**.

Hybrid model in [Frasca et al., 2019]

$y_i \in \mathbb{R}$ agent i's opinion

$$\dot{y}_i = \sum_{j=1}^{n} \frac{a_{ij}}{d_i d_j}(y_j - y_i), \quad (y, a) \in C_{\text{inst}}$$

$$y_i^+ = y_i, \quad (y, a) \in D_{inst}$$

a_{ij} adjacency coefficient

$$a_{ij} = a_{ji} := \begin{cases} 1 & \text{if } i \text{ and } j \text{ interact} \\ 0 & \text{otherwise.} \end{cases}$$

$$\dot{a}_{ij} = 0, \quad (y, a) \in C_{\text{inst}}$$

$$a_{ij}^+ = \begin{cases} a_{hk} & \text{if } (h,k) \neq (i,j) \\ 1 - a_{hk} & \text{if } (h,k) = (i,j), \end{cases} \quad (y, a) \in D_{ij,\text{inst}}^{\text{on}} \cup D_{ij,\text{inst}}^{\text{off}}$$

$$D_{ij,\text{inst}}^{\text{on}} := \left\{ a_{ij} = 0, \ \Gamma_{ij}^{\text{on}}(y, a) \geq \varepsilon \right\},$$

$$D_{ij,\text{inst}}^{\text{off}} := \left\{ a_{ij} = 1, \ \Gamma_{ij}^{\text{off}}(y, a) \leq -\varepsilon \right\},$$

Γ_{ij}^{on} and Γ_{ij}^{off} as a difference between opinion mismatches

$$\Gamma_{ij}^{\text{on}}(y,a) := \overbrace{\sum_{\ell\neq i,\ \ell\neq j}\left[(d_j+1)\frac{a_{i\ell}}{d_i d_\ell}(y_i-y_\ell)^2+(d_i+1)\frac{a_{j\ell}}{d_i d_\ell}(y_j-y_\ell)^2\right]}^{\text{average opinion mismatch between agents i-th and j-th and their respective neighbours}} - \underbrace{\left(1+\frac{\eta^2}{d_i d_j}\right)(y_i-y_j)^2}_{\text{difference between opinions of agents i-th and j-th}}$$

$$D_{ij,\text{inst}}^{\text{on}} := \left\{a_{ij}=0,\ \Gamma_{ij}^{\text{on}}(y,a)\geq\varepsilon\right\}$$

$$\Gamma_{ij}^{\text{off}}(y,a) := \sum_{\ell\neq i,\ \ell\neq j}\left[\frac{d_j a_{i\ell}}{(d_i-1)d_\ell}(y_i-y_\ell)^2+\frac{d_i a_{j\ell}}{(d_j-1)d_\ell}(y_j-y_\ell)^2\right] - \left(1-\frac{\eta^2}{d_i d_j}\right)(y_i-y_j)^2$$

$$D_{ij,\text{inst}}^{\text{off}} := \left\{a_{ij}=1,\ \Gamma_{ij}^{\text{off}}(y,a)\leq-\varepsilon\right\},$$

Memory as a filter of the adaptive thresholds

θ_{ij} as **history of the interaction** between agents i and j.

$$\dot{\theta}_{ij} = -\beta_{ij}\theta_{ij} + (1 - a_{ij})\Gamma^{\text{on}}_{ij}(y, a) + a_{ij}\Gamma^{\text{off}}_{ij}(y, a), \qquad (\star, \theta) \in C_{\text{mem}}$$

If $a_{ij} = 0$, then
$\dot{\theta}_{ij} = -\beta_{ij}\theta_{ij} + \Gamma^{\text{on}}_{ij}(y, a)$

If $a_{ij} = 1$, then
$\dot{\theta}_{ij} = -\beta_{ij}\theta_{ij} + \Gamma^{\text{off}}_{ij}(y, a)$

“Faithful” dynamics $\beta_{ij} \approx 0$

“Forgetful” dynamics $\beta_{ij} \gg 0$

$$\theta^{+}_{ij} = \theta_{ij}, \qquad (\star, \theta) \in D_{mem}$$

Hybrid model with memory-based connectivity

$y_i \in \mathbb{R}$ agent i's opinion

$$\dot{y}_i = \sum_{j=1}^{n} \frac{a_{ij}}{d_i d_j}(y_j - y_i), \quad (\star, \theta) \in C_{\text{mem}}$$

$$y_i^+ = y_i, \quad (\star, \theta) \in D_{mem}$$

$\theta_{ij} \in \mathbb{R}$ history of the interaction

$$\dot{\theta}_{ij} = -\beta_{ij}\theta_{ij} + (1 - a_{ij})\Gamma_{ij}^{\text{on}}(y, a) + a_{ij}\Gamma_{ij}^{\text{off}}(y, a), \quad (\star, \theta) \in C_{\text{mem}}$$

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$$D_{ij,\text{mem}}^{\text{on}} := \left\{ a_{ij} = 0, \ \Gamma_{ij}^{\text{on}}(y, a) \geq \varepsilon, \ \theta_{ij} \geq 0 \right\}, \quad \dot{\theta}_{ij} = -\beta_{ij}\theta_{ij} + \Gamma_{ij}^{\text{on}}(y, a)$$

$$D_{ij,\text{mem}}^{\text{off}} := \left\{ a_{ij} = 1, \ \Gamma_{ij}^{\text{off}}(y, a) \leq -\varepsilon, \ \theta_{ij} \leq 0 \right\} \quad \dot{\theta}_{ij} = -\beta_{ij}\theta_{ij} + \Gamma_{ij}^{\text{off}}(y, a)$$

Memory-based model - $\mathcal{KL}$-stability and clustering

$$\omega(x) := \omega_0(y,a) + \sum_{(i,j)\in\mathcal{E}^+} (1-a_{ij})\max\{0,\theta_{ij}\}$$

$$\omega_0(y,a) := \min_{(z,a)\in\mathcal{A}_{\text{cluster}}} |y-z|, \quad \mathcal{A}_{\text{cluster}} := \{a_{ij}(y_i-y_j)=0\}$$

Theorem

- All maximal solutions to system are **complete and eventually continuous**.
- For each maximal solution x, there exists $x^\star$ such that $x(\mathfrak{t},\mathfrak{j}) \to x^\star$ as $\mathfrak{t}\to\infty$.
- System is $\mathcal{KL}$-**stable with respect to** ω.

Corollary

Each maximal solution converges to a clustering pattern with constant, equal opinions in each cluster.

Continuous opinions and discrete actions in social networks: a multi-agent system approach

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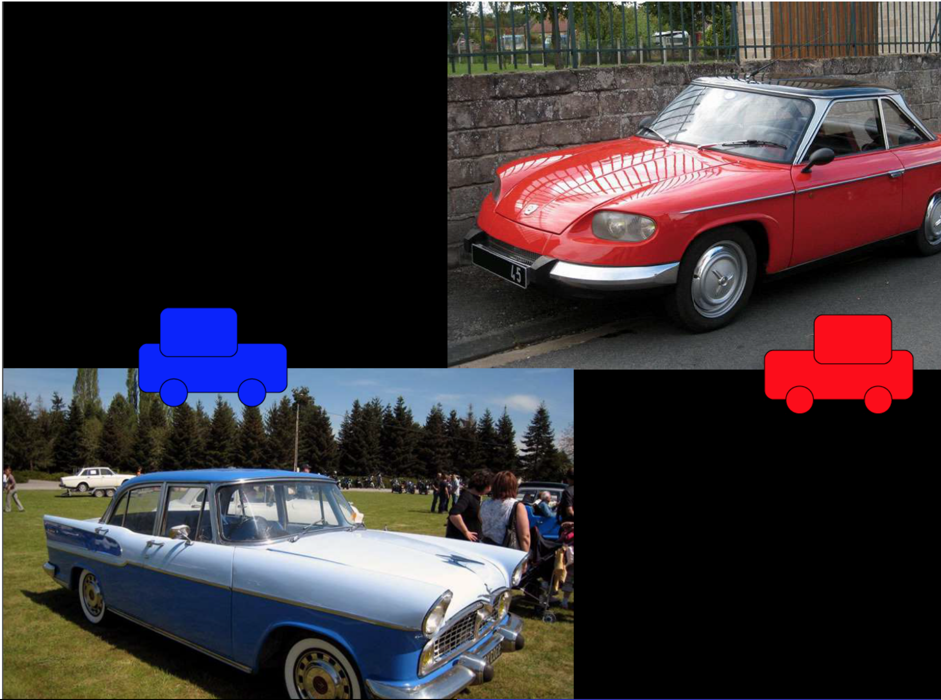
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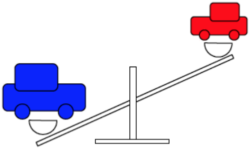
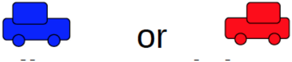
Modeling opinion dynamics

continuous opinions



- DeGroot (consensus)
- Bounded confidence (Hegselmann-Krause, Weisbush-Deffuant)
- Friedkin

discrete opinions

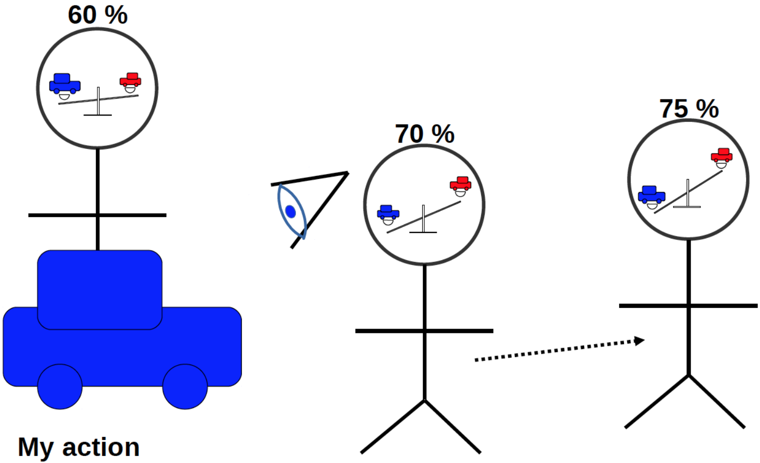


- Ising
- Voter
- Sznajd

CODA Model

A. Martins Int. Journal of Modern Physics C

Continuous opinion but **discrete actions**



CODA Model

Opinion of agent i at time k : $p_i(k) \in [0, 1]$
Action of agent i at time k :

$$q_i(k) = \begin{cases} 0 \text{ if } \left(p_i(k) < \frac{1}{2}\right), \\ 0 \text{ if } \left(p_i(k) = \frac{1}{2} \text{ and } q_i(k-1) = 0\right), \\ 1 \text{ otherwise.} \end{cases}$$

Define

$$ODD(k) = \frac{p_i(k)}{1 - p_i(k)}$$

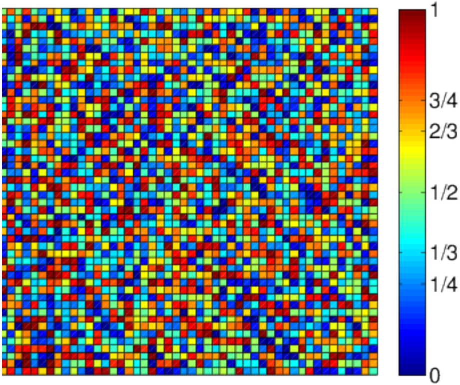
and the update rule

$$ODD(k+1) = \begin{cases} ODD(k) \cdot \dfrac{\alpha}{1-\alpha} \text{ if } q_j(k) = 1 \\ ODD(k) \cdot \dfrac{1-\alpha}{\alpha} \text{ if } q_j(k) = 0 \end{cases}$$

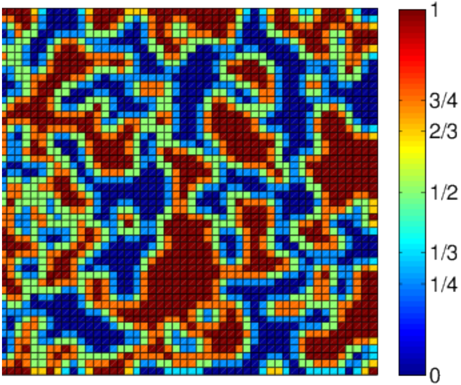
CODA simulation results

$\alpha = 2/3$

initial opinions



final opinions



Can this asymptotic behavior be predicted ?

CODA : a multi-agent modeling

Quantized consensus approximation (Chowdhury et al. CDC 2016)
Opinion of agent i at time k : $p_i(k) \in [0, 1]$
Action of agent i at time k :

$$q_i(k) = \begin{cases} 0 \text{ if } \left(p_i(k) < \frac{1}{2}\right), \\ 0 \text{ if } \left(p_i(k) = \frac{1}{2} \text{ and } q_i(k-1) = 0\right), \\ 1 \text{ otherwise.} \end{cases}$$

Update rule :

$$p_i(k+1) = p_i(k) + \frac{p_i(k)(1 - p_i(k))}{n_i} \sum_{j \in N_i} (q_j(k) - p_i(k)).$$

Comparison of opinion evolution when meeting one neighbor j when $\alpha = 2/3$

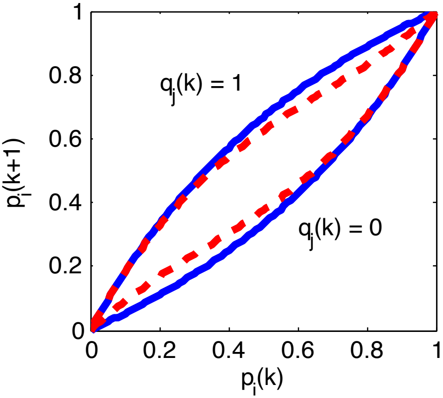


FIGURE – Original CODA, Quantized consensus CODA

Analysis

Basic notation and assumption

Notation : Consider a network of n individuals/agents denoted by $\mathcal{V} = \{1, \ldots, n\}$.

The interaction topology between agents is described by a fixed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ that can be directed or not.

$N^-(k) = \{i \in \mathcal{V} \mid q_i(k) = 0\}$ and $N^+(k) = \{i \in \mathcal{V} \mid q_i(k) = 1\}$

$n^-(k)$ and $n^+(k)$ the cardinality of $N^-(k)$ and $N^+(k)$

$N_i^-(k) = N_i \cap N^-(k)$ and $N_i^+(k) = N_i \cap N^+(k)$

$n_i^-(k)$ and $n_i^+(k)$ the cardinalities of $N_i^-(k)$ and $N_i^+(k)$.

Assumption

There exists a strictly positive constant $\epsilon \in \left(0, \frac{1}{2}\right)$ such that for all $i \in \mathcal{V}$, one has $p_i(0) \in [\epsilon, 1 - \epsilon]$.

Characterization of equilibria

Proposition

A sequence of opinions $\left(p_i(k)\right)_{k\geq 0}$ *converges towards a non-extreme value* $p_i^* \in (0,1)$ *iff the sequence* $\left(n_i^+(k)\right)_{k\geq 0}$ *is stationary. In this case*

$$p_i^* = \lim_{k\to\infty} p_i(k) = \frac{\lim_{k\to\infty} n_i^+(k)}{n_i} \in \mathcal{S}.$$

where

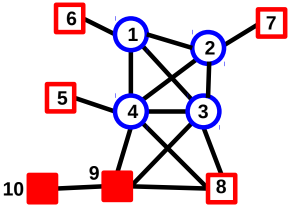
$$\mathcal{S} = \left\{ \frac{k}{m} \;\middle|\; k, m \in \mathbb{N}, k \leq m \leq n-1 \right\} \subset \mathbb{Q}.$$

Robust polarized cluster

Definition

We say a subset of agents $A \subseteq \mathcal{V}$ is a *robust polarized cluster* if the following hold :

- $\forall i, j \in A, \ q_i(0) = q_j(0)$;
- $\forall i \in A, \ |N_i \cap A| \geq |N_i \setminus A|$.



Preservation of actions

Proposition

If $A \subseteq \mathcal{V}$ *is a robust polarized cluster with* $q_i(0) = 0$ *for a certain* $i \in A$ *then*

- $\forall i \in A,\ \forall k \in \mathbb{N},\ q_i(k) = 0\,;$
- $\forall i \in A,\ \lim_{k \to \infty} p_i(k) \leq \frac{1}{2}.$

Propagation of actions

Proposition

Let us consider the sets $A_1, A_2, \ldots, A_d$ *such that*

- A_1 *is a robust polarized set with* $q_i(0) = 0$ *for a certain* $i \in A_1$ *;*
- $\forall h \in \{1, \ldots, d-1\}$ *and* $\forall i \in A_{h+1}$ *one has*

$$|N_i \cap A_h| > |N_i \setminus A_h|.$$

Then, for all $h \in \{1, \ldots, d\}$, $\exists T_h \in \mathbb{N}$, *such that,*

$$\forall k \geq T_h, \forall i \in A_h, q_i(k) = 0,$$

where one can choose $T_1 = 0$. *Consequently,* $\forall i \in \bigcup_{h=1}^{d} A_h$ *one has* $\lim_{k\to\infty} p_i(k) \leq \frac{1}{2}$.

Some particular network topologies

Complete graph

Proposition

If $n^-(0) > n^+(0)$ *then* $\forall i \in \mathcal{V}$ *the limit behavior of the opinion is given by* $\lim_{k\to\infty} p_i(k) = 0$*. Reversely,* $n^+(0) > n^-(0)$ *implies* $\lim_{k\to\infty} p_i(k) = 1$*.*

Proposition

Assume that $n^+(0) = n^-(0)$ *and* $\forall i \in \{1, \ldots, \frac{n}{2}\}$ *there exist* $\eta_i(0) \in \left(0, \frac{1}{2}\right)$ *such that* $p_i(0) = \frac{1}{2} - \eta_i(0)$ *and* $p_{\frac{n}{2}+i}(0) = \frac{1}{2} + \eta_i(0)$*.*
Then $n^+(k) = n^-(k), \ \forall k \in \mathbb{N}$ *and* $\exists k^* \in \mathbb{N}$ *and* ϵ^* *such that* $\forall k \geq k^*, \forall j \in \mathcal{V}$,

$$|p_j(k) - \frac{1}{2}| \leq \epsilon^* \text{ and } \left(p_j(k) - \frac{1}{2}\right)\left(p_j(k+1) - \frac{1}{2}\right) < 0$$

Oscillating behavior

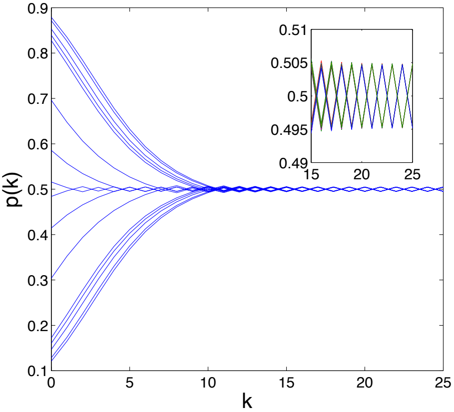


FIGURE – Trajectories of 100 agents with the complete interaction graph and initial opinions distributed symmetrically around $1/2$.

Ring graph

Proposition

Under the ring graph topology the CODA opinions dynamics has to the following properties :

- *the set* $\mathcal{S}$ *reduces to* $\left\{0, \frac{1}{2}, 1\right\}$ *;*
- *if* $\exists i \in \mathcal{V}$ *such that* $q_i(0) = q_{i+1}(0) = 0$ *then* $\{i, i+1\}$ *is a robust polarized cluster (i.e.* $q_i(k) = q_{i+1}(k), \ \forall k \in \mathbb{N}$*);*
- *if* $\forall i \in \mathcal{V}$ *one has* $q_i(0) = 1 - q_{i+1}(0)$ *then*
  - *either the initial opinions are not symmetric w.r.t.* $\frac{1}{2}$ *and agents will change actions asynchronously leading to robust polarized sets* $\{i-1, i, i+1\}$.
  - *or* $p_i(0) \in \left\{\frac{1}{2} - \sigma, \frac{1}{2} + \sigma\right\}, \forall i \in \mathcal{V}$ *and agents will change actions synchronously preserving* $n^-(k) = n^+(k), \forall k \in \mathbb{N}$.

Numerical illustrations

Influential minority

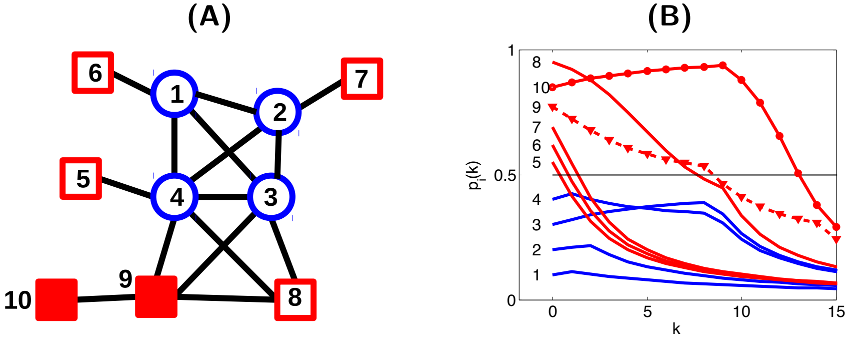
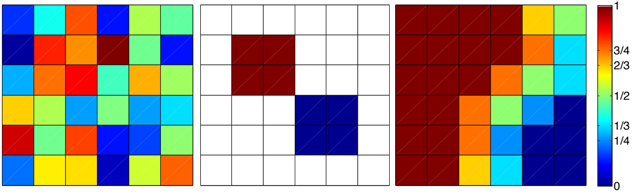
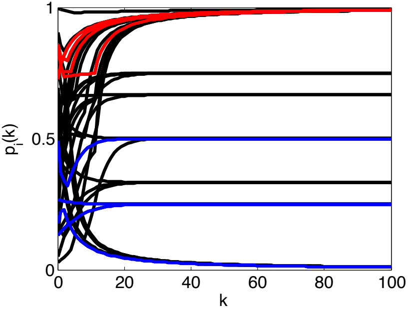


Figure – Illustration of the action diffusion process described in Proposition 3. Agents $1, 2, 3$ and 4 start with $q_i(0) = 0$ and form a robust polarized cluster (Definition 1), while agents $5, 6, 7, 8, 9$ and 10 start with $q_i(0) = 1$. All agents converge to a state with action $\lim q_i = 0$.

CODA on a square lattice





Conclusions and perspectives

Concluding remarks :

- we introduced a novel opinion dynamics model in which agents access only a quantized version of the opinions of their neighbors ;
- the model is able to reproduces real social networks behavior : disensus, clustering, oscillations, opinion propagation ;
- main results provides opinion preservation and opinion propagation ;
- complete analysis for special cases : complete graph, ring graph.

Perspectives :

- extension to updates that take into account a random neighbor ;